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Environmental effects on the geometric phase

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Abstract. The behavior of the geometric phase gained by a single spin-1/2 nucleus immersed into a thermal or a squeezed environment is investigated. Both the time dependence of the phase and its value at infinity are examined against several physical parameters. It is observed that for some intermediate ranges of the temperature and the coupling strength, the presence of squeezing enhances the geometric phase.

1 Introduction

The geometric phase (GP) associated with the paths in the Hilbert space of quantum systems [1–7] has recently attracted a renewed interest due to its possible application in the implementation of quantum gates. As it is purely of geometric origin, it is believed that the GP is robust to the parametric noise that is present in the driving external fields. This intrinsic robustness [8–13] makes it a very good candidate for the implementation of controlled gates [14,15] for fault tolerant quantum computation [16]. These gates are also of crucial importance for a universal set of quantum logic gates [14,15,17]. For this reason, a full understanding of the behavior of the GP, like the behavior of entanglement [18–20], in different environments is vital for quantum computation.

Various aspects of the effects of the environment on the GP of open quantum systems have been studied [21–32]. Rezakhani and Zanardi analyzed the temperature effects on mixed-state GP for a single and two coupled spin-1/2 particles [21]. Wang et al. analyzed the effects of a squeezed vacuum reservoir on GP of a two-level atom in an electromagnetic field by a formulation entirely in terms of geometric structures [22]. Lombardo and Villar studied the time scales at which decoherence becomes effective on GP for environments at different temperatures [23]. Carollo et al. showed that GP can be induced by cyclic evolution in an adiabatically manipulated squeezed vacuum reservoir [24]. Banerjee and Srikanth studied the effects of a squeezed-thermal environment on the GP of a two-level system [25] for dissipative and non-dissipative cases, and analyzed the initial-state and temperature dependence of GP for the system. The purpose of this article is to examine the effect of various physical parameters on the time-dependence of GP.

The system analyzed in this article is a two-level nucleus inside a magnetic field. The time evolution is non-unitary due to the interaction with the environment. The environment is first taken as a thermal bath of bosons. Then, this bath is considered to be driven by an electromagnetic field in a squeezed state in order to see whether the GP can be enhanced. The dependence of the GP on temperature, external magnetic field, coupling strength, squeezing and initial state of the system are then analyzed. The GP is computed by using the kinematic definition given by Tong et al. [33]. In their approach, when the system's density matrix in the Schrödinger picture is $\hat{\rho}(t)$, the GP gained in the time interval $[0,\tau]$ is given by the expression

$$\Phi(\tau) = \arg\left(\sum_k \sqrt{\lambda_k(0)\lambda_k(\tau)}\langle\varphi_k(0)|\varphi_k(\tau)\rangle \times \exp\left\{-\int_0^\tau \langle\varphi_k(t)|\dot{\varphi}_k(t)\rangle dt\right\}\right), \quad (1)$$

where $\lambda_k(t)$ and $|\varphi_k(t)\rangle$ are the eigenvalues and the eigenvectors of $\hat{\rho}(t)$, respectively.

The content of the article is as follows. In Section 2, the system and its interaction with the environment is described. This section also includes the derivation of the non-unitary dynamics of the reduced density matrix by using Markov approximation. After that, the density matrix of the system is computed analytically. In Section 3, the dependence of the GP on various physical parameters is analyzed. Finally, Section 4 contains brief conclusions.

2 The nucleus inside a bath

Our specific system is a single spin-1/2 nucleus in an external static magnetic field $\vec{B}$, which is taken to be in the z direction. The Hamiltonian of the nucleus is

$$\hat{H}_N = -\hbar\omega_N \hat{I}^z, \quad (2)$$

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where $\hbar$ is the Planck constant, $\omega_N = \gamma_N |\vec{B}|$, γ_N is the gyromagnetic ratio of the nucleus, $\hat{I}^z = \hat{\sigma}^z/2$ and $\hat{\sigma}^z$ is the z-component of the Pauli spin operator.

The environment (reservoir) is assumed to be a bath of harmonic oscillators (such as electromagnetic radiation) where the frequency spectrum forms a continuum. The Hamiltonian of the environment is given by

$$\hat{H}_R = \hbar \int_0^\infty \omega \hat{b}^\dagger(\omega)\hat{b}(\omega) d\omega \tag{3}$$

where $\hat{b}(\omega)$ denotes the annihilation operator for the mode at frequency ω. The interaction of nucleus with the oscillators is described by the interaction Hamiltonian,

$$\hat{H}_{NR} = \hbar \int_0^\infty g_N(\omega) \hat{I}^+ \hat{b}^\dagger(\omega) d\omega \; + \text{h.c.}, \tag{4}$$

where $g_N(\omega)$ is the coupling coefficient between the nucleus and the oscillator mode at frequency ω, $\hat{I}^\pm = \hat{I}^x \pm i \hat{I}^y$ are the spin ladder operators, and h.c. indicates the Hermitian conjugated term.

The total Hamiltonian $\hat{H} = \hat{H}_N + \hat{H}_R + \hat{H}_{NR}$, is transformed into the interaction picture as

$$\hat{V}(t) = \hbar \int_0^\infty g_N(\omega) e^{i(\omega-\omega_N)t} \hat{I}^+ \hat{b}^\dagger(\omega) d\omega \; + \text{h.c.} \tag{5}$$

Let $\hat{\rho}_{NR}(t)$ denote the state of the combined system of the nucleus and the environment at time t, and let $\hat{\rho}^I_{NR}(t)$, $\hat{\rho}^I_R(t)$ and $\hat{\rho}^I_N(t)$ denote the states in the interaction picture of the combined system, the environment and the nucleus, respectively. We study this system in the regime in which the Markov approximation can be applied. In this approximation, it is assumed that the environment remains in its initial state during the evolution, $\hat{\rho}^I_R(t) = \hat{\rho}_R(0)$, and the state of the combined system is taken to be in product form, $\hat{\rho}^I_{NR}(t) = \hat{\rho}^I_N(t) \otimes \hat{\rho}_R(0)$. As a result, the equation of motion of the state of the nucleus can be obtained as

$$\dot{\hat{\rho}}^I_N(t) = -\frac{i}{\hbar} \text{tr}_R \left[\hat{V}(t), \hat{\rho}^I_N(0) \otimes \hat{\rho}_R(0)\right] - \frac{1}{\hbar^2} \text{tr}_R \int_0^t \left[\hat{V}(t),\; \left[\hat{V}(t'), \hat{\rho}^I_N(t') \otimes \hat{\rho}_R(0)\right]\right] dt', \tag{6}$$

where the dot denotes the time derivative and tr_R represents the partial trace over the degrees of freedom of the environment [34,35].

For the thermal environment, the state is given by $\hat{\rho}_R(0) = \hat{\rho}_{\text{th}}$ where

$$\hat{\rho}_{\text{th}} = \exp\left(-\hat{H}_R/k_B T\right) / \text{tr} \exp\left(-\hat{H}_R/k_B T\right) \tag{7}$$

is the thermal equilibrium state. If the bath is driven by a squeezed field, the state of the squeezed-thermal environment is given by

$$\hat{\rho}_R(0) = \hat{S} \hat{\rho}_{\text{th}} \hat{S}^\dagger, \tag{8}$$

where the unitary operator $\hat{S}$ is taken as

$$\hat{S} = \hat{S}[\xi(\omega)] = \exp\left(-\frac{1}{2}\int_\omega^{2\Omega} d\omega \left[\xi(\omega)\hat{b}^\dagger(\omega)\hat{b}^\dagger(2\Omega-\omega) - \xi^*(\omega)\hat{b}(2\Omega-\omega)\hat{b}(\omega)\right]\right). \tag{9}$$

Here, 2Ω is the squeezing-carrier frequency and $\xi(\omega) = r(\omega)e^{i\phi(\omega)}$, where $r(\omega)$ and $\phi(\omega)$ are real-valued functions of ω that characterize the squeezing. These functions satisfy the relationship $\xi(\omega) = \xi(2\Omega - \omega)$.

For both types of environments, we have $\langle b(\omega)\rangle = 0$ which makes the first term on the right-hand side of equation (6) vanish. It is shown in Appendix that the second term can be expressed in the form

$$\begin{aligned}\dot{\hat{\rho}}^I_N(t) = &-\frac{i}{\hbar}\Big[\Delta\hat{H}_N, \hat{\rho}^I_N(t)\Big] + C_-\Big(2\hat{I}^-\hat{\rho}^I_N(t)\hat{I}^+ - \hat{\rho}^I_N(t)\hat{I}^+\hat{I}^- \\ &- \hat{I}^+\hat{I}^-\hat{\rho}^I_N(t)\Big) + C_+\Big(2\hat{I}^+\hat{\rho}^I_N(t)\hat{I}^- - \hat{\rho}^I_N(t)\hat{I}^-\hat{I}^+ \\ &- \hat{I}^-\hat{I}^+\hat{\rho}^I_N(t)\Big) - D\,\hat{I}^+\hat{\rho}^I_N(t)\hat{I}^+ - D^*\,\hat{I}^-\hat{\rho}^I_N(t)\hat{I}^-,\end{aligned} \tag{10}$$

where $\Delta\hat{H}_N = -\hbar\Delta\omega_N\hat{\sigma}^z/2$ is an additional term in the Hamiltonian which represents a re-normalization of the frequency ω_N of the nucleus, and $C_\pm$ and D are numbers that capture the collective decohering effects of all of the oscillators in the reservoir. The coefficients $C_\pm$ are independent of time for both models considered in this article and they satisfy

$$C_+ = C_- + \pi |g_N(\omega_N)|^2. \tag{11}$$

The values of these coefficients are computed in Appendix. It appears that $C_\pm$ do not depend on the values of the coupling constant $g_N(\omega)$ for frequencies different from ω_N. However, both D and $\Delta\omega_N$ depend on $g_N(\omega)$ at all frequencies by an expression involving a principal-value integral. Hence, the interaction of the spin with all non-resonant oscillator modes (modes having frequency ω with $\omega \neq \omega_N$) is captured by these two coefficients only. Since there are no further theoretical restrictions on the coupling function $g_N(\omega)$, by adjusting that function it is possible to set all principal-value integrals to various values. The qualitative behavior of the GP does not change much when nonzero quantities are assigned to these integrals. For the sake of simplicity, in this contribution all of these principal-value integrals are taken to be zero and the resultant values of the coefficients are used. With this choice, $\Delta\omega_N$ and $\Delta\hat{H}_N$ becomes zero.

For the purely thermal environment, the constants in equation (10) can be computed as $D = 0$ and $C_- = \pi|g_N(\omega_N)|^2 n(\omega_N)$, where

$$n(\omega) = \frac{1}{\exp(\hbar\omega/k_B T) - 1} \tag{12}$$

denotes the average occupation number of the mode at frequency ω. In this case, the matrix elements of the interaction picture density matrix can be integrated to

$$\rho^I_{N11}(t) = \rho^I_{N11}(\infty) + \left(\rho^I_{N11}(0) - \rho^I_{N11}(\infty)\right) e^{-2(C_+ + C_-)t}, \quad (13)$$

$$\rho^I_{N12}(t) = \rho^I_{N11}(0) e^{-(C_+ + C_-)t}, \quad (14)$$

where

$$\rho^I_{N11}(\infty) = \rho_{N11}(\infty) = \frac{C_+}{C_+ + C_-}. \quad (15)$$

For the squeezed-thermal environment given in equation (8), the coefficients of the equation of motion can be found as

$$C_- = \pi |g_N(\omega_N)|^2 \Big\{ n(\omega_N) + [n(\omega_N) + n(2\Omega - \omega_N) + 1] \times \sinh^2 r(\omega_N) \Big\}, \quad (16)$$

$$D = \pi g_N(\omega_N) g_N(2\Omega - \omega_N) [n(\omega_N) + n(2\Omega - \omega_N) + 1] \times \sinh(2r(\omega_N)) e^{2i(\Omega - \omega_N)t - i\phi(\omega_N)}. \quad (17)$$

For this case too, the time-dependent density matrix can be found analytically. The diagonal entry $\rho^I_{N11}(t)$ is still given by equation (13) and the off-diagonal entry can be expressed in general as

$$\rho^I_{N12}(t) = \left(A_1 e^{st} + A_2 e^{-st}\right) e^{-(C_+ + C_-)t + i(\Omega - \omega_N)t}, \quad (18)$$

where s is the purely real or purely imaginary constant

$$s = \sqrt{|D|^2 - (\Omega - \omega_N)^2}, \quad (19)$$

and A_1 and A_2 are constants that should be determined from the initial state $\hat{\rho}_N(0)$. The squeezing changes the time-dependence of the density matrix as follows. First, it changes the long-time limit of the density matrix $\hat{\rho}_N(\infty)$. It also makes the diagonal relaxation time, $(2(C_+ + C_-))^{-1}$, shorter. Apart from those, it changes the time-dependence of the off-diagonal entry; in particular it produces new oscillatory behavior for sufficiently large squeezing.

The geometric phase $\Phi(\tau)$ given in equation (1) can be computed analytically in the thermal case. The argument of the exponential in this expression is given by

$$-\int_0^\tau \langle \varphi_\pm(t) | \dot{\varphi}_\pm(t) \rangle dt = \mp i \frac{\omega_N}{2} \left(\tau + \frac{1}{C_+ + C_-} (F(\tau) - F(0)) \right) \quad (20)$$

where

$$F(\tau) = \ln(A_\tau - \delta_\tau + 2\delta_\infty) - \ln\left(2(\delta_0 - \delta_\infty) + \frac{|\rho_{N12}(0)|^2}{A_\tau + \delta_\tau} \right), \quad (21)$$

$$A_\tau = \sqrt{\frac{1}{4} - \det \hat{\rho}_N(\tau)}, \quad (22)$$

$$\delta_\tau = \frac{1}{2} \left(\rho_{N11}(\tau) - \rho_{N22}(\tau)\right), \quad (23)$$

and $|\varphi_\pm(\tau)\rangle$ are eigenvectors with corresponding eigenvalues $\lambda_\pm(\tau) = 1/2 \pm A_\tau$.

It should be noted that when τ goes to infinity, the density-matrix $\hat{\rho}_N(\tau)$ goes to a diagonal time-independent state. Hence $F(\tau)$ and consequently the phase $\Phi(\tau)$ tend to constant limits. For the squeezed-thermal environment, the argument of the exponential needs to be computed numerically but it can be shown that the same behavior holds at infinity, i.e., the GP settles down to a well-defined finite limit.

3 The dependence of the geometric phase on physical parameters

For investigating the behavior of the GP under different physical conditions, the principal-value integrals are taken to be zero as explained in the previous section, and both of the coupling coefficients $g_N(\omega_N)$ and $g_N(2\Omega - \omega_N)$ are taken to be equal to a real positive value g. The phase parameter $\phi(\omega_N)$ of the squeezed state is taken to be 0. The relevant parameter $r(\omega_N)$ that gives the amount of squeezing is simply denoted by r below.

In the problem, there are three parameters having the dimension of energy: the excitation energy $\hbar\omega_N$ (which is proportional to the external magnetic field), the thermal energy $k_B T$ and an energy related to the coupling strength $\hbar g^2$. The behavior of the GP is invariant if all three of these parameters are scaled by the same amount. For this reason, only the ratios of these parameters need to be specified as done below.

3.1 Time dependence of the geometric phase

The dependence of the geometric phase $\Phi(\tau)$ on time τ has periodic structures with period equal to $2\pi/\omega_N$. In each of the oscillations over one period, $\Phi(\tau)$ also changes by an amount depending crucially on the values of all physical parameters. If there is squeezing, an additional periodic structure coming from the time-dependence of the off-diagonal entry given in equation (18) may appear.

The effects of changing the temperature, the coupling strength and the magnetic field on the time-dependence of the GP are shown in the Figures 1–3, respectively. As it can be seen, increasing the temperature or the coupling strength, or decreasing the magnetic field have regular effects when there is no squeezing (the figures on the left of Figs. 1–3). With these changes, the oscillations of the GP are destroyed and the limiting value of GP is reached at earlier times.

But when there is squeezing, that regular dependence is lost for the case of increasing the temperature or the coupling strength (Figs. 1 and 2). For sufficiently large values of the temperature or the coupling constant, the limiting value of the GP is higher in comparison to those with lower temperatures or coupling strengths. Moreover the GP reaches the limiting values at a shorter time for higher temperatures and at a longer time for higher coupling

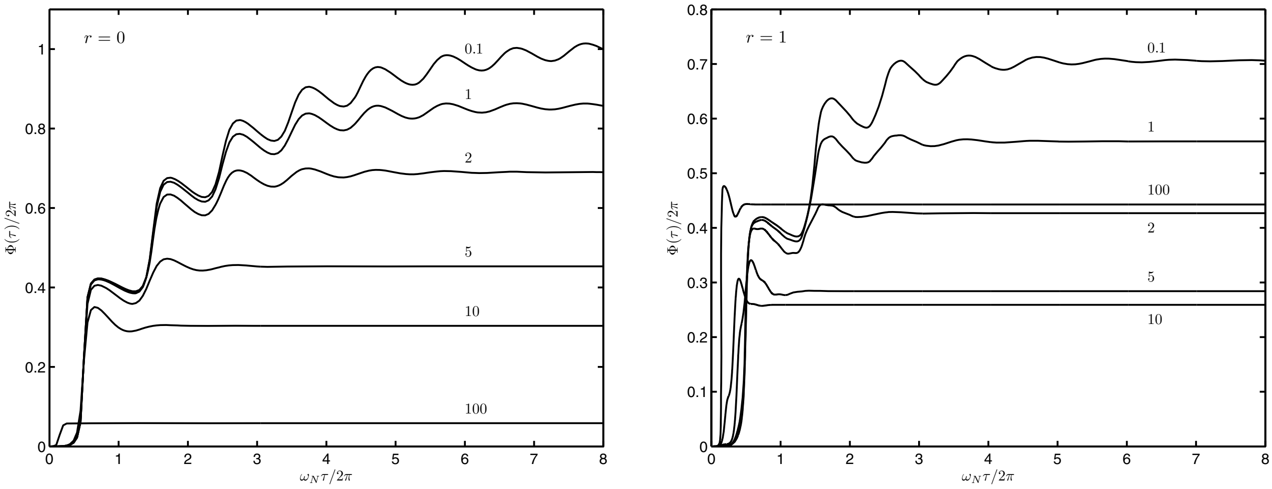

Fig. 1. The GP vs. time for different temperatures for a thermal environment (left) and a squeezed-thermal environment with $r = 1$ (right). The coupling strength is $g^2/\omega_N = 0.01$ and the initial state is a pure spin up state along x. The numbers on the graph indicate the value of $k_B T/\hbar\omega_N$.

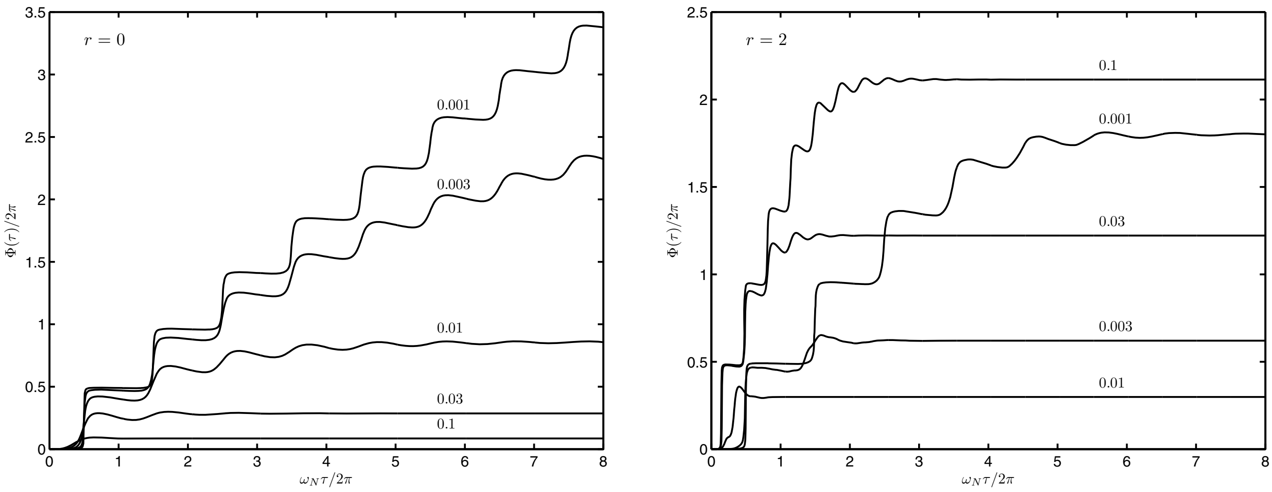

Fig. 2. The GP vs. time for different coupling strengths for a thermal environment (left) and a squeezed-thermal environment with $r = 2$ (right). The magnetic field is $\hbar\omega_N/k_B T = 1$ and the initial state is a pure spin up state along x. The values of the ratio g^2/ω_N are shown on the graphs. As can be seen, squeezing enhances the GP for a particular case with large coupling. Note also small oscillations which are produced by the effect of squeezed field on the spin state.

strengths. For low temperatures and coupling strengths, presence of squeezing in general decreases the GP.

For high magnetic fields, the squeezing has almost no effect (Fig. 3). At lower fields, the squeezing tends to decrease the GP, destroys the oscillations and make the limiting values reached at earlier times. And for small enough field strengths it is possible to get higher values for the GP with increasing the squeezing.

Figure 4 shows the dependence of the GP on the initial state. The initial state is taken as a pure state along a direction on the xz-plane having the spherical angle θ. When θ is increased above $\pi/2$, the GP changes significantly. Especially, the GP tends to decrease at the initial moments for $\theta > \pi/2$. If the initial state is mixed, then the GP is suppressed in magnitude but its overall behavior does not change.

3.2 The dependence of the phase at infinity on physical parameters

The behavior of the GP at infinity is also a quantity of interest. Its magnitude gives an indication of the overall magnitudes of the GP that can be obtained at any time. This value can also be used for showing the effect of the squeezing on the GP. In Figure 5, the long time limit of the GP, i.e., $\Phi(\infty)$, is shown as a function of temperature. When there is no squeezing, increasing the temperature always suppresses the GP. With the presence of squeezing, three different temperature regimes appear. For very low and very high temperatures, the GP is still suppressed with increasing T. However, there is now an intermediate range of temperatures over which the opposite trend appears. Increasing the squeezing parameter r moves this

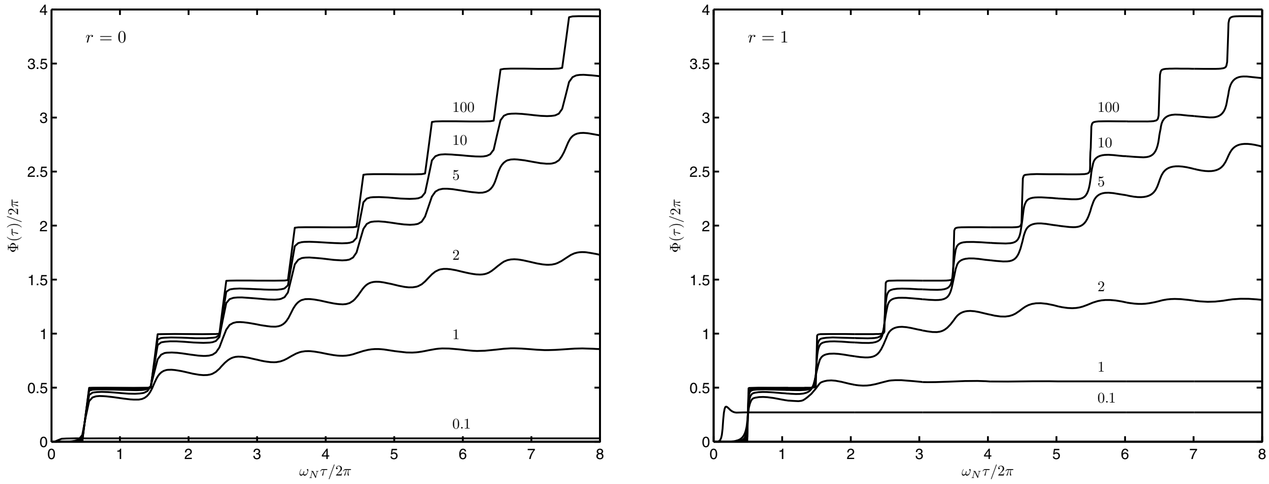

Fig. 3. The GP vs. time for different magnetic fields for a thermal environment (left) and a squeezed-thermal environment with $r = 1$ (right). The coupling coefficient is taken as $\hbar g^2/k_B T = 0.01$ and the initial state is a pure spin up state along x. The values of the ratio $\hbar\omega_N/k_B T$ are shown on the graph. We have taken $\Omega = 3\omega_N$ for all data points.

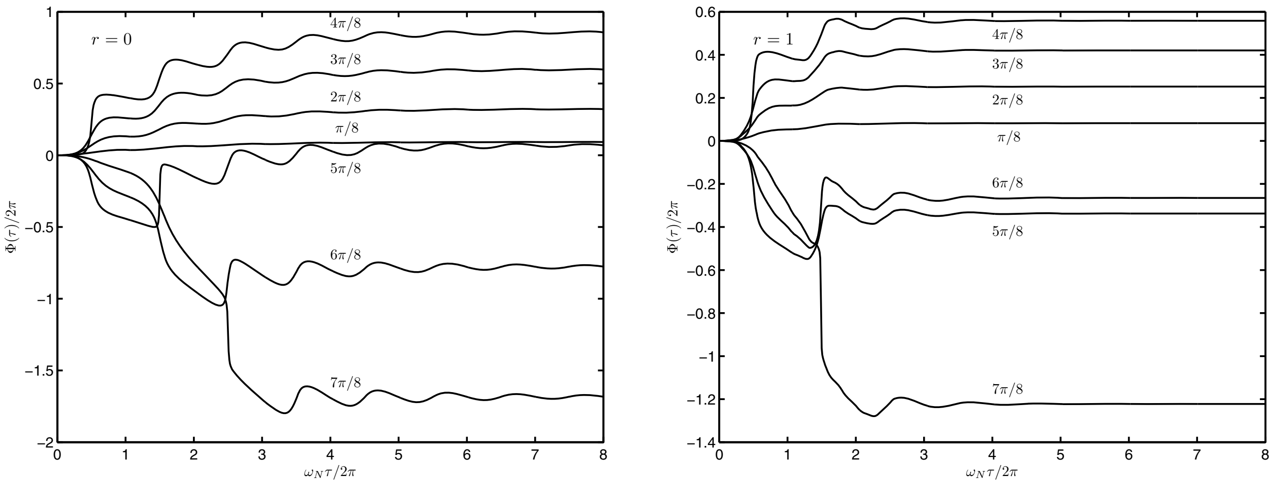

Fig. 4. The GP vs. time for different initial states for a thermal environment (left) and a squeezed-thermal environment with $r = 1$ (right). The magnetic field is $\hbar\omega_N/k_B T = 1$ and the coupling strength is $g^2/\omega_N = 0.01$. The initial state is chosen as a pure state and the value of the spherical angle θ of the initial orientation is shown on the graph.

intermediate range to lower temperatures. Figure 6 shows the dependence of the limiting value of the GP on the coupling strength. The behavior is similar to the case of changing the temperature parameter.

Figure 7 shows the dependence of the limiting value of the GP on the magnetic field. It can be seen that there are two regions for this case. For high enough magnetic fields this dependence is almost linear. Squeezing essentially decreases the slope. But at lower magnetic fields, GP has a different behavior under squeezing. In this regime, squeezing enhances the GP which is now a nonlinear function of the magnetic field. The limiting GP also reaches to a maximum value at an optimum value of the magnetic field. Increasing the squeezing expands that region of nonlinearity to higher magnetic fields.

The dependence of $\Phi(\infty)$ on the initial state is shown in Figure 8. As can be seen, for low squeezing $\Phi(\infty)$ reaches its largest values for θ near π, i.e., a spin orientation which is almost spin down. Small squeezing suppresses the GP but does not change the behavior. For large squeezing however, $\Phi(\infty)$ can be enhanced significantly and its largest values are reached when θ is around $\pi/2$.

4 Conclusion

The availability of easy manipulation with the current technology NMR techniques makes nuclear systems good candidates as carrier systems of the entities that are necessary for quantum information processing [14,36–40]. Such a system is recently tested by Cucchietti et al.[1] [41]. With this motivation, the effects of the coupling between a nucleus and a dissipative environment on the GP that the

[1] The tested system in this experiment is a spin-1/2 system coupled to an environment near a quantum phase transition.

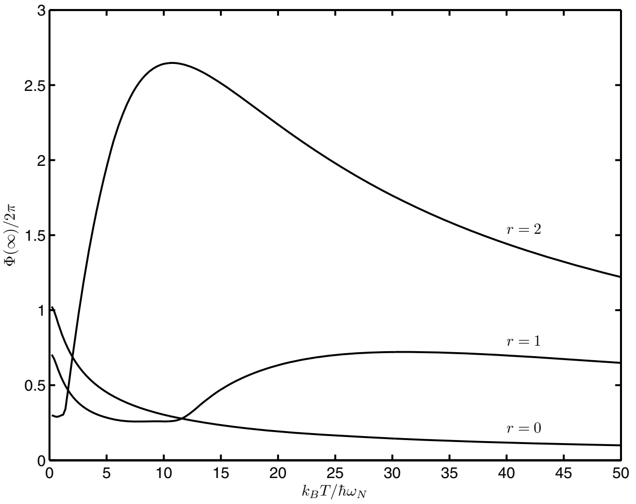

Fig. 5. The GP at infinity as a function of temperature for different values of the squeezing parameter r. The coupling strength satisfies $g^2/\omega_N = 0.01$.

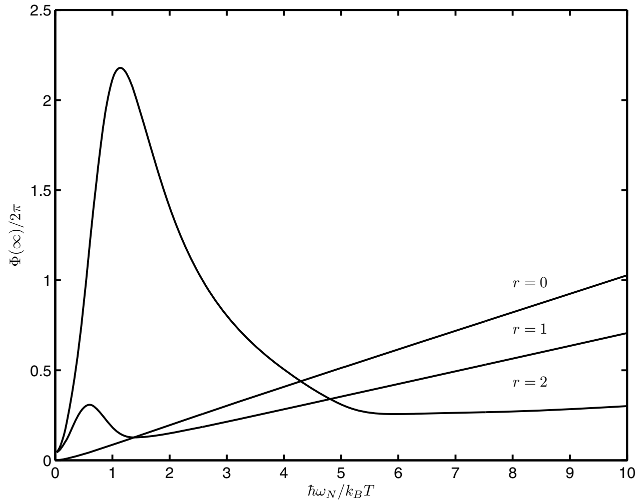

Fig. 7. The GP at infinity as a function of the magnetic field for different values of the squeezing parameter r. The coupling strength satisfies $\hbar g^2/k_B T = 0.1$.

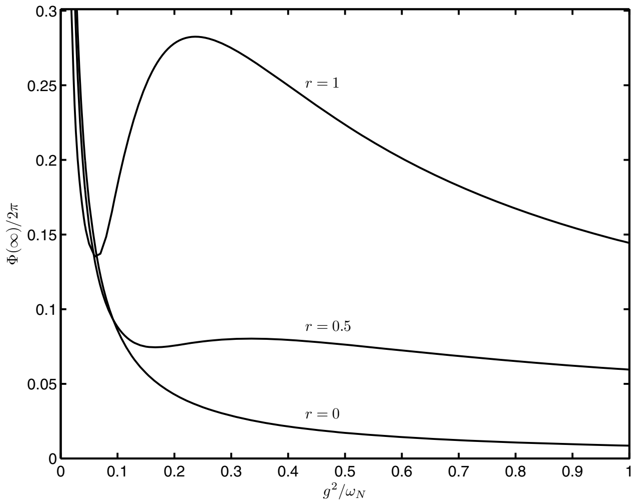

Fig. 6. The GP at infinity as a function of the coupling strength for different values of the squeezing parameter r for the case $\hbar\omega_N/k_B T = 1$.

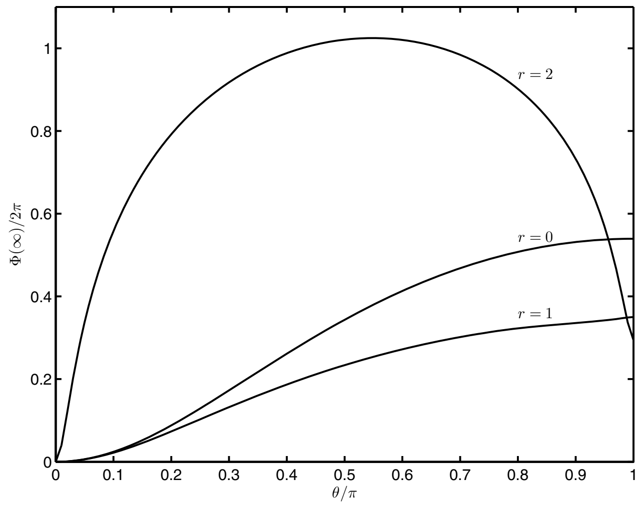

Fig. 8. The GP at infinity as a function of the initial state angle θ for different values of the squeezing parameter r. The parameters are chosen as $g^2/\omega_N = 0.025$ and $\hbar\omega_N/k_B T = 1$.

state of the nucleus gains during its time evolution are studied. Although it is affected by the physical parameters, it is always possible to get a finite GP. Its dependence on these parameters are almost as expected; it decreases with increasing temperature or coupling strength, because both increase the decoherence caused by the environment, and it increases with increasing magnetic field, since increasing B has the same effect as decreasing both the temperature and the coupling constant.

If the GP needs to be increased further, the environment could be driven by an electromagnetic field in a squeezed state. However, in this case, one should be very careful because the squeezing may have a destructive or an enhancing effect on the GP depending on the parameters. In order to enhance the GP, the temperature should be held at some appropriate intermediate value; it should not be too small or too large. The coupling strength should be taken at a relatively high value so that the squeezed field can interact with the spin more; but it should not be too large lest the spin relaxes to its equilibrium state quickly. And larger magnetic fields do not always enhance the GP; it is true only up to an upper bound for the field strength. Even though the external magnetic fields are time-dependent in the actual implementations of the quantum gates, the dependence of GP on all of the external parameters will qualitatively be the same as above. Hence, the geometric phase of a system with parameters chosen as described above will be robust enough to be used in quantum computation.

At the very early stage of this work, we have had useful discussions with (late) Prof. A.S. Shumovsky, for which we are thankful.

Appendix: Derivation of the equation of motion

To derive equation (10) starting from equation (6) we first note that the first term of equation (6) drops for both models of the environment that are used in this article. For the second term, we change the integration variable by $t' = t - \tau$ and then extend the upper bound of the τ-integral to infinity while ignoring the variations of nucleus' state in the integrand. Thus, equation (6) becomes

$$\dot{\hat{\rho}}_N^I(t) = -\frac{1}{\hbar^2}\int_0^\infty d\tau \mathrm{tr}_R\left[\hat{V}(t),\left[\hat{V}(t-\tau),\hat{\rho}_N^I(t)\otimes\hat{\rho}_R(0)\right]\right]. \tag{A.1}$$

The interaction picture Hamiltonian in equation (5) can be expressed as the sum of two terms

$$\hat{V}(t) = \hbar\sum_{\mu=\pm}\hat{I}^\mu\otimes\hat{B}_\mu(t) \tag{A.2}$$

where

$$\hat{B}_+(t) = \int d\omega g_N(\omega)e^{i(\omega-\omega_N)t}\hat{b}^\dagger(\omega) \tag{A.3}$$

and $\hat{B}_-(t) = \hat{B}_+^\dagger(t)$. Inserting $\hat{V}^\dagger(t-\tau) = \hat{V}(t-\tau)$ to the integral above, we get

$$\dot{\hat{\rho}}_N^I(t) = \sum_{\mu,\nu=\pm}\gamma_{\mu\nu}(t)\left(\hat{I}^{\nu\dagger}\hat{\rho}_N^I(t)\hat{I}^\mu - \hat{I}^\mu\hat{I}^{\nu\dagger}\hat{\rho}_N^I(t)\right) + \sum_{\mu,\nu=\pm}\gamma'_{\mu\nu}(t)\left(\hat{I}^\mu\hat{\rho}_N^I(t)\hat{I}^{\nu\dagger} - \hat{\rho}_N^I(t)\hat{I}^{\nu\dagger}\hat{I}^\mu\right) \tag{A.4}$$

where

$$\gamma_{\mu\nu}(t) = \int_0^\infty d\tau\langle B_\mu(t)B_\nu^\dagger(t-\tau)\rangle, \tag{A.5}$$

$$\gamma'_{\mu\nu}(t) = \int_0^\infty d\tau\langle B_\nu^\dagger(t-\tau)B_\mu(t)\rangle. \tag{A.6}$$

Note that these two integrals are related by

$$\gamma'_{\mu,\nu}(t) = \gamma^*_{-\mu,-\nu}(t), \tag{A.7}$$

a relation which will be used in simplifying the equations further.

Some terms can be shuffled around in equation (A.4) in order to bring it into the standard Lindbladian form as

$$\dot{\hat{\rho}}_N^I(t) = \sum_{\mu,\nu=\pm}\gamma_{\mu\nu}(t)\left(\hat{I}^{\nu\dagger}\hat{\rho}_N^I(t)\hat{I}^\mu - \frac{1}{2}\hat{I}^\mu\hat{I}^{\nu\dagger}\hat{\rho}_N^I(t) - \hat{\rho}_N^I(t)\frac{1}{2}\hat{I}^\mu\hat{I}^{\nu\dagger}\right) + \sum_{\mu,\nu=\pm}\gamma'_{\mu\nu}(t)\left(\hat{I}^\mu\hat{\rho}_N^I(t)\hat{I}^{\nu\dagger} - \frac{1}{2}\hat{\rho}_N^I(t)\hat{I}^{\nu\dagger}\hat{I}^\mu - \frac{1}{2}\hat{\rho}_N^I(t)\hat{I}^{\nu\dagger}\hat{I}^\mu\right) - \frac{i}{\hbar}\left[\Delta\hat{H}_N,\hat{\rho}_N^I(t)\right] \tag{A.8}$$

where

$$\Delta\hat{H}_N = \frac{i\hbar}{2}\sum_{\mu\nu}\left(\gamma'_{\mu\nu}(t)\hat{I}^{\nu\dagger}\hat{I}^\mu - \gamma_{\mu\nu}(t)\hat{I}^\mu\hat{I}^{\nu\dagger}\right) = \frac{i\hbar}{2}\sum_{\mu\nu}\left(\gamma^*_{\nu\mu}(t) - \gamma_{\mu\nu}(t)\right)\hat{I}^\mu\hat{I}^{\nu\dagger}. \tag{A.9}$$

The last line is obtained by invoking the relation in equation (A.7) and the fact that $\hat{I}^{\pm\dagger} = \hat{I}^\mp$. Same relations can be utilized to simplify the equation of motion for the state of the nucleus as

$$\dot{\hat{\rho}}_N^I(t) = \frac{1}{2}\sum_{\mu,\nu=\pm}\left(\gamma_{\mu\nu}(t) + \gamma^*_{\nu\mu}(t)\right) \times\left(2\hat{I}^{\nu\dagger}\hat{\rho}_N^I(t)\hat{I}^\mu - \hat{I}^\mu\hat{I}^{\nu\dagger}\hat{\rho}_N^I(t) - \hat{\rho}_N^I(t)\hat{I}^\mu\hat{I}^{\nu\dagger}\right) - \frac{i}{\hbar}\left[\Delta\hat{H}_N,\hat{\rho}_N^I(t)\right]$$

which is simply equation (10).

For the squeezed environment described by the state equation (8), the following expectation values are needed for computing the coefficients of the equation of motion.

$$\left\langle b(\omega)b^\dagger(\omega')\right\rangle = \left[(n(\omega)+1)\cosh^2 r(\omega) + n(2\Omega-\omega)\times\sinh^2 r(\omega)\right]\delta(\omega-\omega') \tag{A.10}$$

$$\left\langle b^\dagger(\omega')b(\omega)\right\rangle = \left\langle b(\omega)b^\dagger(\omega')\right\rangle - \delta(\omega-\omega') \tag{A.11}$$

$$\left\langle b(\omega)b(\omega')\right\rangle = -\frac{1}{2}\left[n(\omega)+n(2\Omega-\omega)+1\right]\sinh 2r(\omega)\times e^{i\phi(\omega)}\delta(\omega+\omega'-2\Omega). \tag{A.12}$$

Using these, the coefficients $\gamma_{\mu\nu}(t)$ can be computed as follows.

$$\gamma_{++}(t) = i\int d\omega|g_N(\omega)|^2\times\frac{n(\omega)\cosh^2 r(\omega) + (n(2\Omega-\omega)+1)\sinh^2 r(\omega)}{\omega-\omega_N+i\epsilon} \tag{A.13}$$

$$\gamma_{--}(t) = -i\int d\omega|g_N(\omega)|^2\times\frac{(n(\omega)+1)\cosh^2 r(\omega) + n(2\Omega-\omega)\sinh^2 r(\omega)}{\omega-\omega_N-i\epsilon} \tag{A.14}$$

$$\gamma_{+-}(t) = \frac{i}{2}e^{2i(\Omega-\omega_N)t}\int d\omega g_N(\omega)g_N(2\Omega-\omega)\times\frac{\left[n(\omega)+n(2\Omega-\omega)+1\right]\sinh 2r(\omega)e^{-i\phi(\omega)}}{\omega-\omega_N-i\epsilon} \tag{A.15}$$

and $\gamma_{-+}(t) = \gamma^*_{+-}(t)$.

The coefficients of equation (10) are given by

$$C_\mp = \mathrm{Re}\gamma_{\pm,\pm}(t), \quad D = -2\gamma_{+-}(t). \tag{A.16}$$

It can be seen that C_- is given exactly by equation (16) and the relation in equation (11) is satisfied. The expression for D is more complicated because the principal value integral in $\gamma_{+-}(t)$ has a non-zero contribution; equation (17) only gives the contribution coming from the Dirac-delta term of the integral of $\gamma_{+-}(t)$. Note, however, that the time dependence of D will be unaffected by the inclusion of principal value integral. Finally, the re-normalization of the Hamiltonian is

$$\varDelta H_N = \hbar\left(\mathrm{Im}\gamma_{++}\hat{I}^+\hat{I}^- + \mathrm{Im}\gamma_{--}\hat{I}^-\hat{I}^+\right). \quad \text{(A.17)}$$

Dropping the dynamically irrelevant term proportional to the identity, it can be seen that $\varDelta H_N$ re-normalizes the frequency ω_N in equation (2) by

$$\varDelta\omega_N = \mathrm{Im}\left(\gamma_{--} - \gamma_{++}\right) \quad \text{(A.18)}$$

$$= -\mathcal{P}\int d\omega |g_N(\omega)|^2 \times \frac{2n(\omega)+1+2\left[n(\omega)+n(2\varOmega-\omega)+1\right]\sinh^2 r(\omega)}{\omega-\omega_N} \quad \text{(A.19)}$$

where $\mathcal{P}$ represents principal value.

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