EXTENDED JACOBI MATRIX POLYNOMIALS

RABİA AKTAŞ*, BAYRAM ÇEKİM AND ALİ ÇEVİK

ABSTRACT. In this paper, extended Jacobi matrix polynomials (EJMPs) are introduced. The matrix differential equation satisfied by them is given. A Rodrigues formula, orthogonality, linear generating matrix functions and recurrence relations are presented for these matrix polynomials. Furthermore, general families of multilinear and multilateral generating matrix functions are obtained and their applications are presented.

1. INTRODUCTION

Special matrix functions seen on statistics, Lie group theory and number theory are well known in [4, 19]. In the recent papers, the classical orthogonal polynomials have been extended to the orthogonal matrix polynomials [2, 5, 7, 11, 12, 14, 16]. Jódar and Cortés introduced and studied the hypergeometric matrix function $F(A, B; C; z)$ and the hypergeometric matrix differential equation in [14] and the explicit closed form general solution of it has been given in [13]. In [3, 5, 6, 10, 11, 16], Chebyshev, Gegenbauer, Laguerre and Hermite matrix polynomials were introduced and various results were given for these matrix polynomials. In [7], Defez et al. introduced and studied Jacobi matrix polynomials so that $P_n^{(A,B)}(x)$ for parameter matrices A and B whose eigenvalues, z, all satisfy $Re(z) > -1$. For any natural number $n \geq 0$, the nth Jacobi matrix polynomial $P_n^{(A,B)}(x)$ is defined by

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{n!} F\left(A + B + (n+1)I, -nI; B + I; \frac{1+x}{2}\right)(B + I)_n.$$

Also, Defez et al. shows that these matrix polynomials have the Rodrigues formula:

$$P_n^{(A,B)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-A}(1+x)^{-B}\frac{d^n}{dx^n}\left[(1-x)^{A+nI}(1+x)^{B+nI}\right] \tag{1.1}$$

and satisfy the following orthogonality relation [7]:

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* Corresponding author.

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$$\int_{-1}^{1} (1-x)^A (1+x)^B P_n^{(A,B)}(x) P_m^{(A,B)}(x)\, dx \tag{1.2}$$

$$= \begin{cases} \frac{2^{A+B+I}}{n!}\Gamma(A+B+(2n+1)I) \\ .\Gamma^{-1}(A+B+(n+1)I)\Gamma(B+(n+1)I) & , \quad m=n \\ .\Gamma(A+(n+1)I)\Gamma^{-1}(A+B+2(n+1)I) \\ \\ \mathbf{0} & , \quad m \neq n \end{cases}$$

where A and $B \in \mathbb{C}^{r\times r}$ satisfy

$$\mathrm{Re}(z) > -1 \text{ for } z \in \sigma(A),\ \mathrm{Re}(\eta) > -1 \text{ for } \eta \in \sigma(B),\ AB = BA.$$

In the scalar case, in order to give an unified presentation of the classical orthogonal polynomials (especially Jacobi, Laguerre and Hermite polynomials), Fujiwara [9] studied the polynomial $F_n^{(\alpha,\beta)}(x;a,b,c)$ called extended Jacobi polynomial (EJP) and defined by the Rodrigues formula

$$\begin{aligned} F_n^{(\alpha,\beta)}(x;a,b,c) &= \frac{(-c)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta} \\ &\cdot \frac{d^n}{dx^n}\left\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\right\},\ (c>0). \end{aligned} \tag{1.3}$$

The polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ are essentially those that were considered by Szegö himself [18, p. 58], who showed (by means of a simple linear transformation) that these polynomials are just a constant multiple of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. By comparing the Rodrigues representation for classical Jacobi polynomials and (1.3), it is not difficult to rewrite Szegö's observation [18, p. 58, equation (4.1.2)] in the form (cf., e.g., [17, p. 388, Problem 11],[1],[15]):

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \{c(a-b)\}^n P_n^{(\alpha,\beta)}\left(\frac{2(x-a)}{a-b}+1\right) \tag{1.4}$$

or, equivalently,

$$P_n^{(\alpha,\beta)}(x) = \{c(a-b)\}^{-n} F_n^{(\alpha,\beta)}\left(\frac{1}{2}\{a+b+(a-b)x\};a,b,c\right). \tag{1.5}$$

Thus, as already pointed out by Srivastava and Manocha [17], the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ may be looked upon as being equivalent to (and not as a generalization of) the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Furthermore, by recourse to certain limiting processes, it is easily verified that the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ would give rise to the Laguerre and Hermite polynomials (and indeed also the Bessel polynomials) just as the classical

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ do. Consequently, the main purpose of Fujiwara's investigation [9] is already served by the classical Jacobi polynomials themselves.

The EJPs $F_n^{(\alpha,\beta)}(x;a,b,c)$ are orthogonal over the interval (a,b) with respect to the weight function $\omega(x;a,b) = (x-a)^\alpha (b-x)^\beta$. In fact, it is hold that

$$\int_a^b (x-a)^\alpha (b-x)^\beta F_n^{(\alpha,\beta)}(x;a,b,c) F_m^{(\alpha,\beta)}(x;a,b,c)\,dx \tag{1.6}$$

$$= \frac{c^{m+n}(-1)^{\alpha+\beta+1}(a-b)^{m+n+\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}\delta_{m,n}$$

where $\delta_{m,n}$ is Kronecker delta and $\min\{\mathbb{R}(\alpha), \mathbb{R}(\beta)\} > -1;\ m,n \in \mathbb{N}_0 := \mathbb{N}\cup\{0\} = \{0,1,2,...\}$.

Throughout this paper, for a matrix $A \in \mathbb{C}^{r\times r}$, its spectrum is denoted by $\sigma(A)$. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$\|A\| = \sup_{x\neq 0}\frac{\|Ax\|_2}{\|x\|_2},$$

where, for a vector $y \in \mathbb{C}^r$, $\|y\|_2 = (y^T y)^{1/2}$ is the Euclidean norm of y. I and $\mathbf{0}$ will denote the identity matrix and the null matrix in $\mathbb{C}^{r\times r}$, respectively. We say that a matrix A in $\mathbb{C}^{r\times r}$ is a positive stable if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A. If $A_0, A_1, ..., A_n$ are elements of $\mathbb{C}^{r\times r}$ and $A_n \neq \mathbf{0}$, then we call

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0$$

a matrix polynomial of degree n in x. From [14], one can see

$$(A)_n = A(A+I)(A+2I)...(A+(n-1)I);\ n \geq 1\ ;\ (A)_0 = I. \tag{1.7}$$

From the relation (1.7), we see that

$$\frac{(-1)^k}{(n-k)!}I = \frac{(-nI)_k}{n!}\ ;\ 0 \leq k \leq n. \tag{1.8}$$

The hypergeometric matrix function $F(A,B;C;z)$ has been given in the form [14]

$$F(A,B;C;z) = \sum_{n=0}^{\infty}\frac{(A)_n(B)_n}{n!}\left[(C)_n\right]^{-1} z^n$$

for matrices A, B and C in $\mathbb{C}^{r\times r}$ such that $C+nI$ is invertible for all integer $n \geq 0$ and for $|z| < 1$. For any matrix A in $\mathbb{C}^{r\times r}$, the authors exploited the following relation due to [14]

$$(1-x)^{-A} = \sum_{n=0}^{\infty}\frac{(A)_n}{n!}x^n\ ,\ |x| < 1. \tag{1.9}$$

In [8], if $f(z)$ and $g(z)$ are holomorphic functions in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{r\times r}$ for which $\sigma(A) \subset \Omega$, then

$$f(A)g(A) = g(A)f(A).$$

Hence, if $B \in \mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B) \subset \Omega$ and $AB = BA$, then

$$f(A)g(B) = g(B)f(A).$$

Furthermore, in [7], the reciprocal scalar Gamma function, $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z. Thus, for any $C \in \mathbb{C}^{r\times r}$, the Riesz-Dunford functional calculus [8] shows that $\Gamma^{-1}(C)$ is well defined and is, indeed, the inverse of $\Gamma(C)$. Hence: if $C \in \mathbb{C}^{r\times r}$ is such that $C + nI$ is invertible for every integer $n \geq 0$, then

$$(C)_n = \Gamma(C + nI)\Gamma^{-1}(C).$$

2. Extended Jacobi matrix polynomials and their some properties

In this section, we define extended Jacobi matrix polynomials (EJMPs) $F_n^{(A,B)}(x; a, b, c)$ and give some properties satisfied by these polynomials.

Definition 2.1. *Let A and $B \in \mathbb{C}^{r\times r}$ be matrices satisfying the spectral conditions $\mathrm{Re}(z) > -1$ for each eigenvalue $z \in \sigma(A)$ and $\mathrm{Re}(\eta) > -1$ for each eigenvalue $\eta \in \sigma(B)$. For any natural number $n \geq 0$, extended Jacobi matrix polynomials of degree n are defined by*

$$\begin{aligned} F_n^{(A,B)}(x; a, b, c) &= (c(b-a))^n \\ &\cdot \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^{n+k}}{(b-a)^k n!} \Gamma(A + B + (n + k + 1) I) \\ &.\Gamma^{-1}(A + B + (n + 1) I)\Gamma(A + (n + 1) I)\Gamma^{-1}(A + (k + 1) I)(x - a)^k \\ &(c > 0). \end{aligned} \tag{2.1}$$

With the help of hypergeometric matrix function, the matrix polynomials $F_n^{(A,B)}(x; a, b, c)$ defined by (2.1) can be rewritten as follows:

Remark 2.1. *For EJMPs, we have*

$$\begin{aligned} &(c(a-b))^{-n} n! F_n^{(A,B)}(x; a, b, c) \\ = \ &F\left(-nI, A + B + (n + 1) I; A + I; \frac{x-a}{b-a}\right)(A + I)_n . \end{aligned} \tag{2.2}$$

Theorem 2.1. *The extended Jacobi matrix polynomials*

$$Y(x) = F_n^{(A,B)}(x; a, b, c)$$

satisfy the matrix differential equation:

$$(x-a)(b-x)Y''(x)+\{a(A+B+I)+bI-(A+B+2I)x\}Y'(x) \\ +(b-a)Y'(x)A+n(A+B+(n+1)I)Y(x)=\mathbf{0}\ ,\ a<x<b \tag{2.3}$$

where all eigenvalues z of the matrices A and B satisfy the condition $\mathrm{Re}(z)>-1$.

Proof. The hypergeometric matrix function $W(z)=F(\alpha,\beta;\gamma;z)$ satisfies matrix differential equation [13]

$$z(1-z)W''(z)-z\alpha W'(z)+W'(z)(\gamma-z(\beta+I))-\alpha W(z)\beta=\mathbf{0} \tag{2.4}$$

where $0<|z|<1$, $\alpha,\beta,\gamma\in\mathbb{C}^{r\times r}$, $\gamma\beta=\beta\gamma$ and also $(\gamma+nI)$ is invertible for all positive integer $n\geq 0$. Getting $z=\dfrac{x-a}{b-a}$, $\alpha=A+B+(n+1)I$, $\beta=-nI$ and $\gamma=A+I$, we have

$$\begin{aligned} & F\left(-nI,A+B+(n+1)I;A+I;\frac{x-a}{b-a}\right)(A+I)_n \\ = & \ (c(a-b))^{-n}\,n!F_n^{(A,B)}(x;a,b,c) \end{aligned}$$

from Remark 2.1. Using differential equation (2.4), we obtained the desired differential equation. □

Corollary 2.2. *In (2.3), getting $a=-1,b=1$ and $c=\frac{1}{2}$, we obtain Jacobi matrix differential equation for the polynomials $P_n^{(B,A)}(x)$ [7]. For the scalar case $r=1$, taking $A=\alpha$ and $B=\beta$ with $\alpha,\beta>-1$ in (2.3) gives the scalar extended Jacobi differential equation. If we get $A=\alpha$ and $B=\beta$, with $\alpha,\beta>-1$, $a=-1,b=1$ and $c=\frac{1}{2}$ in (2.3), then (2.3) is reduced to scalar differential equation satisfied by the Jacobi polynomials $P_n^{(\beta,\alpha)}(x)$.*

Corollary 2.3. *The polynomial $Y(x)=F_n^{(A,B)}(x;a,b,c)$ is a solution of the differential equation*

$$\begin{aligned} & \frac{d}{dx}\left[(x-a)(b-x)^{A+B+I}Y'(x)\left(\frac{x-a}{b-x}\right)^{A}\right] \\ & +n(A+B+(n+1)I)(b-x)^{A+B}Y(x)\left(\frac{x-a}{b-x}\right)^{A}=\mathbf{0} \end{aligned} \tag{2.5}$$

for $a<x<b$.

Proof. Premultiplying (2.3) by $(b-x)^{A+B}$ and postmultiplying it by $\left(\dfrac{x-a}{b-x}\right)^{A}$ and rearranging, we have (2.5) for $a<x<b$. □

The following lemma derived in [5] will be useful for obtaining Rodrigues formula for EJMPs.

Lemma 2.4. *(Defez and Jódar,[5]) For C and $D \in \mathbb{C}^{r\times r}$, suppose that D is positively stable, $DC = CD$, and that $C - D + kI$ and $C + kI$ are invertible for all nonnegative integers k. Then, for $|t| < 1$,*

$$F(-nI, D; C; t) = (1-t)^n F\left(-nI, C-D; C; \frac{-t}{1-t}\right), \quad n = 0, 1, \ldots. \tag{2.6}$$

Theorem 2.5. *Let A and B be matrices in $\mathbb{C}^{r\times r}$ satisfying the spectral conditions $\mathrm{Re}(z) > -1$ for each eigenvalue $z \in \sigma(A)$ and $\mathrm{Re}(\eta) > -1$ for each eigenvalue $\eta \in \sigma(B)$ and $AB = BA$. Then, the extended Jacobi matrix polynomials defined by (2.1) have the following Rodrigues formula:*

$$\begin{aligned} F_n^{(A,B)}(x; a, b, c) &= \frac{(-c)^n}{n!}(x-a)^{-A}(b-x)^{-B} \\ &\quad \cdot \frac{d^n}{dx^n}\left[(x-a)^{A+nI}(b-x)^{B+nI}\right], \quad (c>0) \end{aligned} \tag{2.7}$$

for $n = 0, 1, 2, \ldots$.

Proof. Let $D = (A + B + (n+1)I)$ and $C = A + I$ denote in Lemma 2.4 and let D be positively stable. Then $DC = CD$ and the other hypotheses of Lemma 2.4 are satisfied. Therefore, by (2.6)

$$\begin{aligned} & F\left(-nI, A+B+(n+1)I; A+I; \frac{x-a}{b-a}\right) \\ &= \left(\frac{b-x}{b-a}\right)^n F\left(-nI, -(B+nI); A+I; \frac{x-a}{x-b}\right). \end{aligned}$$

If we substitute this equality in (2.2) and use (1.8), we have

$$\begin{aligned} F_n^{(A,B)}(x; a, b, c) &= \frac{(-c)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}(-(B+nI))_k\left[(A+I)_k\right]^{-1} \\ &\quad .(x-a)^k(b-x)^{n-k}(A+I)_n. \end{aligned} \tag{2.8}$$

Moreover,

$$\begin{aligned} \frac{d^{n-k}}{dx^{n-k}}\left[(x-a)^{A+nI}\right] &= \left[(A+I)_k\right]^{-1}(A+I)_n(x-a)^A(x-a)^k, \\ \frac{d^k}{dx^k}\left[(b-x)^{B+nI}\right] &= (-(B+nI))_k(b-x)^B(b-x)^{n-k}, \end{aligned}$$

so, (2.8) can be rewritten in the form

$$F_n^{(A,B)}(x;a,b,c) = \frac{(-c)^n}{n!}(x-a)^{-A}(b-x)^{-B}$$

$$\cdot \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}\left[(b-x)^{B+nI}\right] \frac{d^{n-k}}{dx^{n-k}}\left[(x-a)^{A+nI}\right].$$

Therefore, we obtain

$$\begin{aligned} & F_n^{(A,B)}(x;a,b,c) \\ = & \frac{(-c)^n}{n!}(x-a)^{-A}(b-x)^{-B}\frac{d^n}{dx^n}\left[(x-a)^{A+nI}(b-x)^{B+nI}\right], \end{aligned}$$

which completes the proof. □

By comparing the Rodrigues representations (1.1) and (2.7), we can write the following result:

Corollary 2.6. *The extended Jacobi matrix polynomials* $F_n^{(A,B)}(x;a,b,c)$ *are a constant multiple of Jacobi matrix polynomials* $P_n^{(A,B)}(x)$ *in the form*

$$F_n^{(A,B)}(x;a,b,c) = (c(a-b))^n P_n^{(A,B)}\left(\frac{2(x-a)}{a-b}+1\right). \tag{2.9}$$

By means of (1.2) and (2.9), one can easily see the next theorem for EJMPs.

Theorem 2.7. *Let* A *and* B *be matrices in* $\mathbb{C}^{r\times r}$ *satisfying the spectral conditions* $\mathrm{Re}(z) > -1$ *for each eigenvalue* $z \in \sigma(A)$ *and* $\mathrm{Re}(\eta) > -1$ *for each eigenvalue* $\eta \in \sigma(B)$ *and* $AB = BA$. *Then,*

$$\int_a^b (x-a)^A (b-x)^B F_n^{(A,B)}(x;a,b,c) F_m^{(A,B)}(x;a,b,c)\,dx$$

$$= \begin{cases} \begin{array}{l} \frac{c^{2n}}{n!}(b-a)^{A+B+(2n+1)I}\Gamma(A+B+(2n+1)I) \\ .\Gamma^{-1}(A+B+(n+1)I)\Gamma(B+(n+1)I) \\ .\Gamma(A+(n+1)I)\Gamma^{-1}(A+B+2(n+1)I) \end{array} & , \quad m=n \\ \mathbf{0} & , \quad m \neq n \end{cases}$$

for $m,n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Remark 2.2. *In Theorem 2.7, getting* $a=-1, b=1$ *and* $c=\frac{1}{2}$ *gives the relation (1.2) for the polynomials* $P_n^{(B,A)}(x)$ *[7]. For the scalar case* $r=1$, *if we take* $A=\alpha$ *and* $B=\beta$ *with* $\alpha,\beta > -1$ *in Theorem 2.7, we have (1.6). Getting* $A=\alpha$ *and* $B=\beta$ *, with* $\alpha,\beta>-1$, $a=-1, b=1$ *and* $c=\frac{1}{2}$ *gives orthogonality relation for the classical Jacobi polynomials* $P_n^{(\beta,\alpha)}(x)$.

3. Generating matrix functions for the EJMPs

In this section, we derive families of linear generating matrix functions for the extended Jacobi matrix polynomials.

Theorem 3.1. *Assume that all eigenvalues z of the matrices A and B of the extended Jacobi matrix polynomials $F_n^{(A,B)}(x;a,b,c)$ satisfy condition $Re(z) > -1$. Then we have*

$$\sum_{n=0}^{\infty}(c(a-b))^{-n}(A+B+I)_n\, F_n^{(A,B)}(x;a,b,c)\left[(A+I)_n\right]^{-1}t^n$$

$$=(1-t)^{-(A+B+I)}F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};A+I;\frac{4t(x-a)}{(a-b)(1-t)^2}\right), \tag{3.1}$$

where $|t|<1$ and $A,B\in\mathbb{C}^{r\times r}$.

Proof. By (1.8) and (2.2), we easily see that

$$\sum_{n=0}^{\infty}(c(a-b))^{-n}(A+B+I)_n\, F_n^{(A,B)}(x;a,b,c)\left[(A+I)_n\right]^{-1}t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left\{\frac{(A+B+I)_n(-nI)_k(A+B+(n+1)I)_k}{n!k!}\right.$$
$$\left.\cdot\left[(A+I)_k\right]^{-1}\left(\frac{x-a}{b-a}\right)^k t^n\right\}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left\{\frac{(A+B+I)_n(-1)^k(A+B+(n+1)I)_k}{(n-k)!\,k!}\right.$$
$$\left.\cdot\left[(A+I)_k\right]^{-1}\left(\frac{x-a}{b-a}\right)^k t^n\right\}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(A+B+I)_{n+2k}\,(-1)^k}{n!k!}\left[(A+I)_k\right]^{-1}\left(\frac{x-a}{b-a}\right)^k t^{n+k}$$

$$=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{(A+B+I)_{2k}\,(-1)^k t^n}{n!}\left(\frac{x-a}{b-a}\right)^k\right.$$
$$\left.\cdot\frac{(A+B+(2k+1)I)_n\,t^k}{k!}\left[(A+I)_k\right]^{-1}\right\}$$

where

$$(A+B+I)_n(A+B+(n+1)I)_k=(A+B+I)_{n+k}.$$

By (1.9), since

$$\sum_{n=0}^{\infty} \frac{(A+B+(2k+1)\,I)_n\; t^n}{n!} = (1-t)^{-(A+B+(2k+1)I)},$$

we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} (c\,(a-b))^{-n}\,(A+B+I)_n\; F_n^{(A,B)}\,(x;a,b,c)\,[(A+I)_n]^{-1}\,t^n \\
= \;& \sum_{k=0}^{\infty} \left\{ \frac{(A+B+I)_{2k}\;(-1)^k(1-t)^{-(A+B+(2k+1)I)}t^k}{k!} \right. \\
& \left. .\,[(A+I)_k]^{-1} \left(\frac{x-a}{b-a}\right)^k \right\}.
\end{aligned}$$

From (1.7), we may write that

$$(A+B+I)_{2k} = 2^{2k} \left(\frac{A+B+I}{2}\right)_k \left(\frac{A+B+2I}{2}\right)_k,$$

which implies

$$\begin{aligned}
&\sum_{n=0}^{\infty} (c\,(a-b))^{-n}\,(A+B+I)_n\; F_n^{(A,B)}\,(x;a,b,c)\,[(A+I)_n]^{-1}\,t^n \\
= \;& (1-t)^{-(A+B+I)} \sum_{k=0}^{\infty} \left\{ \frac{2^{2k}\left(\frac{A+B+I}{2}\right)_k \left(\frac{A+B+2I}{2}\right)_k\;(-1)^k t^k}{k!(1-t)^{2k}} \right. \\
& \left. .\,[(A+I)_k]^{-1} \left(\frac{x-a}{b-a}\right)^k \right\} \\
= \;& (1-t)^{-(A+B+I)} F\left(\frac{A+B+I}{2}, \frac{A+B+2I}{2}; A+I;\; \frac{4t(x-a)}{(a-b)\,(1-t)^2}\right)
\end{aligned}$$

The proof is completed. □

In a similar manner as in the proof of Theorem 3.1, one can easily obtain the next results.

Theorem 3.2. *Assume that all eigenvalues z of the matrices A and B of the extended Jacobi matrix polynomials $F_n^{(A,B)}\,(x;a,b,c)$ satisfy condition $Re(z) > -1$. Then, we have*

$$\sum_{n=0}^{\infty}(c\,(a-b))^{-n}\,(A+B+I)_n\,F_n^{(A,B)}\,(x;a,b,c)\left[(B+I)_n\right]^{-1}t^n$$
$$=(1+t)^{-(A+B+I)}\,F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};B+I;\frac{4t(x-b)}{(a-b)\,(1+t)^2}\right), \tag{3.2}$$

where $|t|<1$ *and* $A,B\in\mathbb{C}^{r\times r}$.

Theorem 3.3. *Let* $A,B,C,D\in\mathbb{C}^{r\times r}$. *We get*

$$\begin{aligned}&\sum_{n=0}^{\infty}(c\,(a-b))^{-n}\,F_n^{(A,B)}\,(x;a,b,c)\,t^n\\ =\;&F_4\left(I+B,I+A;I+A,I+B;\frac{(x-a)\,t}{a-b},\frac{(x-b)\,t}{a-b}\right),\end{aligned}$$

where $AB=BA$ *and* $F_4\,(A,B;C,D;x,y)$ *is the matrix version of the Appell's function of two variables which is defined by*

$$\begin{aligned}F_4\,(A,B;C,D;x,y)\;&=\;\sum_{n,k=0}^{\infty}(A)_{n+k}\,(B)_{n+k}\,(D)_n^{-1}\,(C)_k^{-1}\,\frac{x^ky^n}{k!n!}\\ &\qquad\left(\sqrt{x}+\sqrt{y}<1\right),\end{aligned}$$

where $C+nI$ *and* $D+nI$ *are invertible for every integer* $n\geq 0$ *(see* [2]*).*

Theorem 3.4. *Let* $A,B,C,D\in\mathbb{C}^{r\times r}$. *We have*

$$\begin{aligned}&\sum_{n=0}^{\infty}(C)_n(D)_n(I+B)_n^{-1}\,(c\,(a-b))^{-n}\,F_n^{(A,B)}\,(x;a,b,c)\,(I+A)_n^{-1}t^n\\ =\;&F_4\left(C,D;I+A,I+B;\frac{(x-a)\,t}{a-b},\frac{(x-b)\,t}{a-b}\right).\end{aligned}$$

4. Multilinear and multilateral generating matrix functions for EJMPs

In this section, we derive several families of bilinear and bilateral generating matrix functions for the extended Jacobi matrix polynomials generated by (3.1) and given explicitly by (2.1).

We first state our result as the following.

Theorem 4.1. *Corresponding to a non-vanishing function* $\Omega_\mu(y_1,\ldots,y_s)$ *consisting of* s *complex variables* $y_1,\ldots,y_s$ $(s\in\mathbb{N})$ *and of complex order* μ, *let*

$$\Lambda_{\mu,\nu}(y_1,\ldots,y_s;z):=\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(y_1,\ldots,y_s)z^k \qquad (a_k\neq 0,\;\;\mu,\nu\in\mathbb{C}) \tag{4.1}$$

and

$$\Theta_{n,p,\mu,\nu}(x;y_1,...,y_s;\zeta) := \sum_{k=0}^{[n/p]} \Big\{ a_k \left(c\left(a-b\right)\right)^{-n+pk} (A+B+I)_{n-pk}$$

$$\cdot\; F_{n-pk}^{(A,B)}(x;a,b,c) \left[(A+I)_{n-pk}\right]^{-1} \Omega_{\mu+\nu k}(y_1,...,y_s) \Big\} \zeta^k \tag{4.2}$$

where $A, B \in \mathbb{C}^{r\times r}$, $n, p \in \mathbb{N}$ *and (as usual)* $[\lambda]$ *represents the greatest integer in* $\lambda \in \mathbb{R}$. *Then we have*

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x;y_1,...,y_s;\frac{\eta}{t^p}\right) t^n \tag{4.3}$$

$$= (1-t)^{-(A+B+I)} F\left(\frac{A+B+I}{2}, \frac{A+B+2I}{2}; A+I; \frac{4t(x-a)}{(a-b)(1-t)^2}\right)$$

$$.\Lambda_{\mu,\nu}(y_1,...,y_s;\eta),$$

$$(|t| < 1)$$

provided that each member of (4.3) *exists.*

Proof. For convenience, let S denote the first member of the assertion (4.3) of Theorem 4.1. Then, upon substituting for the polynomials

$$\Theta_{n,p,\mu,\nu}\left(x;y_1,...,y_s;\frac{\eta}{t^p}\right)$$

from the definition (4.2) into the left-hand side of (4.3), we obtain

$$S = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{[n/p]} a_k \left(c\left(a-b\right)\right)^{-n+pk} (A+B+I)_{n-pk} F_{n-pk}^{(A,B)}(x;a,b,c) \right.$$

$$\left. \cdot \left[(A+I)_{n-pk}\right]^{-1} \Omega_{\mu+\nu k}(y_1,...,y_s)\eta^k t^{n-pk} \right\}. \tag{4.4}$$

Upon inverting the order of summation in (4.4), if we replace n by $n+pk$, we can write

$$S = \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{\infty} a_k \left(c\left(a-b\right)\right)^{-n} (A+B+I)_n\ F_n^{(A,B)}(x;a,b,c) \right.$$

$$\left. \cdot \left[(A+I)_n\right]^{-1} \Omega_{\mu+\nu k}(y_1,...,y_s)\eta^k t^n \right\}$$

$$= \left\{ \sum_{n=0}^{\infty} \left(c\left(a-b\right)\right)^{-n} (A+B+I)_n\ F_n^{(A,B)}(x;a,b,c) \left[(A+I)_n\right]^{-1} t^n \right.$$

$$\left. \cdot \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1,...,y_s)\eta^k \right\}$$

$$= (1-t)^{-(A+B+I)} F\left(\frac{A+B+I}{2}, \frac{A+B+2I}{2}; A+I; \frac{4t(x-a)}{(a-b)(1-t)^2}\right)$$

$$.\Lambda_{\mu,\nu}(y_1,...,y_s;\eta),$$

which completes the proof of Theorem 4.1. $\square$

By expressing the multivariable function

$$\Omega_{\mu+\nu k}(y_1, \ldots, y_s) \ (k \in \mathbb{N}_0, \ s \in \mathbb{N})$$

in terms of simpler function of one and more variables, we can give further applications of Theorem 4.1. For example, if we set

$$s = 1 \text{ and } \Omega_{\mu+\nu k}(y) = L^{(C,\lambda)}_{\mu+\nu k}(y)$$

in Theorem 4.1, where the nth Laguerre matrix polynomial $L^{(A,\lambda)}_n(x)$ is defined by [11]

$$L^{(A,\lambda)}_n(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!\,(n-k)!} (A+I)_n \left[(A+I)_k\right]^{-1} x^k,$$

where A is a matrix in $\mathbb{C}^{r\times r}$, $A + nI$ is invertible for every integer $n \geq 0$ and λ is a complex number with $\mathrm{Re}(\lambda) > 0$ and generated by

$$\sum_{n=0}^{\infty} L^{(A,\lambda)}_n(x) t^n = (1-t)^{-(A+I)} \exp\left(\frac{-\lambda x t}{1-t}\right), \tag{4.5}$$

$$|t| < 1, \ 0 < x < \infty,$$

then we obtain the following result which provides a class of bilateral generating matrix functions for the extended Jacobi matrix polynomials and the Laguerre matrix polynomials.

Corollary 4.2. *Let* $\Lambda_{\mu,\nu}(y; z) := \sum\limits_{k=0}^{\infty} a_k L^{(C,\lambda)}_{\mu+\nu k}(y) z^k$ *where*

$$a_k \neq 0, \ \mu, \nu \in \mathbb{N}_0$$

and

$$\begin{aligned}\Theta_{n,p,\mu,\nu}(x; y; \zeta) \ :&= \sum_{k=0}^{[n/p]} \Big\{ a_k \left(c\,(a-b)\right)^{-n+pk} (A+B+I)_{n-pk} \\ &\quad .\, F^{(A,B)}_{n-pk}(x; a, b, c) \left[(A+I)_{n-pk}\right]^{-1} L^{(C,\lambda)}_{\mu+\nu k}(y) \zeta^k \Big\}\end{aligned}$$

where $n, p \in \mathbb{N}$. *Then we have*

$$\begin{aligned}&\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x; y; \frac{\eta}{t^p}\right) t^n = (1-t)^{-(A+B+I)} \\ &.F\left(\frac{A+B+I}{2}, \frac{A+B+2I}{2}; A+I; \frac{4t(x-a)}{(a-b)(1-t)^2}\right) \Lambda_{\mu,\nu}(y; \eta)\end{aligned} \tag{4.6}$$

provided that each member of (4.6) *exists.*

Remark 4.1. *Using the generating relation* (4.5) *for the Laguerre matrix polynomials and taking* $a_k = 1$, $\mu = 0$, $\nu = 1$, *we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{[n/p]}(c\,(a-b))^{-n+pk}\,(A+B+I)_{n-pk}\right.\\
&\left.\cdot\,F_{n-pk}^{(A,B)}(x;a,b,c)\left[(A+I)_{n-pk}\right]^{-1}L_k^{(C,\lambda)}(y)\eta^k t^{n-pk}\right\}\\
=\;&(1-t)^{-(A+B+I)}F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};A+I;\frac{4t(x-a)}{(a-b)\,(1-t)^2}\right)\\
&\cdot\,(1-\eta)^{-(C+I)}\exp\left(\frac{-\lambda y\eta}{1-\eta}\right),
\end{aligned}$$

where $|t| < 1$, $|\eta| < 1$, $0 < y < \infty$.

Set

$$s = 1 \text{ and } \Omega_{\mu+\nu k}(y) = (A+B+I)_k\,P_{\mu+\nu k}^{(A,B)}(y)\left[(B+I)_k\right]^{-1}$$

in Theorem 4.1, where the nth Jacobi matrix polynomial $P_n^{(A,B)}(x)$ is defined by Defez et al. in [7] and generated by

$$\begin{aligned}
&\sum_{n=0}^{\infty}(A+B+I)_n\,P_n^{(A,B)}(x)\left[(B+I)_n\right]^{-1}t^n\\
=\;&(1+t)^{-(A+B+I)}\,F\left(\frac{A+B+I}{2},\frac{A+B+2I}{2};B+I;\frac{2t(1+x)}{(1+t)^2}\right),\\
&(|t|<1) \qquad (4.7)
\end{aligned}$$

where all eigenvalues of matrices A and B satisfy $Re(z) > -1$, which was given in [2]. Then, we obtain the following result which provides a class of bilateral generating matrix functions for the extended Jacobi matrix polynomials and the Jacobi matrix polynomials.

Corollary 4.3. *Let* $\Lambda_{\mu,\nu}(y;z) := \sum_{k=0}^{\infty} a_k(C+D+I)_k\,P_{\mu+\nu k}^{(C,D)}(y)(D+I)_k^{-1}z^k$ *where* ($a_k \neq 0$, $\mu,\nu \in \mathbb{N}_0$); *and*

$$\begin{aligned}
&\Theta_{n,p,\mu,\nu}(x;y;\zeta)\\
:=\;&\sum_{k=0}^{[n/p]}\left\{a_k\,(c\,(a-b))^{-n+pk}\,(A+B+I)_{n-pk}\,F_{n-pk}^{(A,B)}(x;a,b,c)\right.\\
&\left.\cdot\left[(A+I)_{n-pk}\right]^{-1}(C+D+I)_k\,P_{\mu+\nu k}^{(C,D)}(y)(D+I)_k^{-1}\zeta^k\right\}
\end{aligned}$$

where $n,p \in \mathbb{N}$ *and all eigenvalues* z *of the matrices* C *and* D *satisfy the condition* $Re(z) > -1$. *Then we have*

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x; y; \frac{\eta}{t^p}\right) t^n = (1-t)^{-(A+B+I)}$$

$$.F\left(\frac{A+B+I}{2}, \frac{A+B+2I}{2}; A+I; \frac{4t(x-a)}{(a-b)(1-t)^2}\right) \Lambda_{\mu,\nu}(y;\eta) \qquad (4.8)$$

provided that each member of (4.8) *exists.*

Remark 4.2. *Using the generating relation* (4.7) *for the Jacobi matrix polynomials and taking* $a_k = 1$, $\mu = 0$, $\nu = 1$, *we have*

$$\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{[n/p]} (c(a-b))^{-n+pk} (A+B+I)_{n-pk}\, F_{n-pk}^{(A,B)}(x;a,b,c) \right.$$

$$\left. \cdot\, [(A+I)_{n-pk}]^{-1} (C+D+I)_k\, P_k^{(C,D)}(y)(D+I)_k^{-1} \eta^k t^{n-pk} \right\}$$

$$= (1-t)^{-(A+B+I)} F\left(\frac{A+B+I}{2}, \frac{A+B+2I}{2}; A+I; \frac{4t(x-a)}{(a-b)(1-t)^2}\right)$$

$$.(1+\eta)^{-(C+D+I)}\, F\left(\frac{C+D+I}{2}, \frac{C+D+2I}{2}; D+I; \frac{2\eta(1+y)}{(1+\eta)^2}\right),$$

where $|\eta| < 1$ *and* $|t| < 1$.

Choose $s = 1$ and $\Omega_{\mu+\nu k}(y) = F_{\mu+\nu k}^{(A,B)}(y;a,b,c)$ in Theorem 4.1 where $\mu, \nu \in \mathbb{N}_0$. We obtain the following class of bilinear generating matrix functions for the extended Jacobi matrix polynomials.

Corollary 4.4. *Let* $\Lambda_{\mu,\nu}(y;z) := \sum_{k=0}^{\infty} a_k F_{\mu+\nu k}^{(C,D)}(y;a,b,c)\, z^k$ *where*

$$a_k \neq 0, \ \ \mu, \nu \in \mathbb{N}_0$$

and

$$\Theta_{n,p,\mu,\nu}(x;y;\zeta) \ :=\ \sum_{k=0}^{[n/p]} \left\{ a_k (c(a-b))^{-n+pk} (A+B+I)_{n-pk} \right.$$

$$\left. \cdot\, F_{n-pk}^{(A,B)}(x;a,b,c)\, [(A+I)_{n-pk}]^{-1}\, F_{\mu+\nu k}^{(C,D)}(y;a,b,c)\, \zeta^k \right\}$$

where $n, p \in \mathbb{N}$ *and all eigenvalues* z *of the matrices* C *and* D *satisfy the condition* $Re(z) > -1$. *Then we have*

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x; y; \frac{\eta}{t^p}\right) t^n = (1-t)^{-(A+B+I)}$$

$$.F\left(\frac{A+B+I}{2}, \frac{A+B+2I}{2}; A+I; \frac{4t(x-a)}{(a-b)(1-t)^2}\right) \Lambda_{\mu,\nu}(y;\eta) \qquad (4.9)$$

provided that each member of (4.9) *exists.*

Remark 4.3. *Using Theorem 3.3 and taking* $a_k = (c(a-b))^{-k}$, $\mu = 0$, $\nu = 1$, *we have*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{[n/p]} (c(a-b))^{-n+pk} (A+B+I)_{n-pk} \, F_{n-pk}^{(A,B)}(x;a,b,c) \right. \\
&\left. \cdot \left[(A+I)_{n-pk}\right]^{-1} (c(a-b))^{-k} F_k^{(C,D)}(y;a,b,c) \, \eta^k t^{n-pk} \right\} \\
= \; & (1-t)^{-(A+B+I)} F\left(\frac{A+B+I}{2}, \frac{A+B+2I}{2}; A+I; \frac{4t(x-a)}{(a-b)(1-t)^2} \right) \\
& . F_4\left(I+D, I+C; I+C, I+D; \frac{(y-a)\eta}{a-b}, \frac{(y-b)\eta}{a-b} \right)
\end{aligned}$$

where $CD = DC$ *and* $\sqrt{\frac{(y-a)\eta}{a-b}} + \sqrt{\frac{(y-b)\eta}{a-b}} < 1$.

Furthermore, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable function $\Omega_{\mu+\nu k}(y_1, ..., y_s)$, ($s \in \mathbb{N}$), is expressed as an appropriate product of several simpler functions, the assertions of Theorem 4.1 can be applied in order to derive various families of multilinear and multilateral generating matrix functions for the extended Jacobi matrix polynomials.

5. Some Recurrence Relations for EJMPs

In this section, some recurrence relations satisfied by EJMPs are given.

Theorem 5.1. *Let* A *and* B *be matrices in* $\mathbb{C}^{r\times r}$ *whose eigenvalues,* z*, all satisfy* $Re(z) > -1$*. EJMPs satisfy*

(i)
$$\begin{aligned}
&\frac{d}{dx} F_n^{(A,B)}(x;a,b,c) \\
&= c\,(A+B+(n+1)\,I)\, F_{n-1}^{(A+I,B+I)}(x;a,b,c)
\end{aligned}$$

(ii)
$$\begin{aligned}
&\frac{d^k}{dx^k} F_n^{(A,B)}(x;a,b,c) \\
&= c^k\,(A+B+(n+1)\,I)_k \; F_{n-k}^{(A+kI,B+kI)}(x;a,b,c)
\end{aligned}$$

for $0 \le k \le n$.

Proof. **(i)** By using (2.2), it can be proved.
(ii) It is enough to use **(i)**. □

In order to obtain some recurrence relations, we use the following theorem derived in [2].

Theorem 5.2. *Assume that $\Psi(u)$ has the formal power-series expansion*

$$\Psi(u)=\sum_{n=0}^{\infty}\gamma_n u^n\ ,\ \gamma_0\neq\mathbf{0}\ ,\ \gamma_n\in\mathbb{C}^{r\times r}. \tag{5.1}$$

Define the matrix polynomials $f_n(x)$ by

$$(1-t)^{-C}\ \Psi\left(\frac{-4xt}{(1-t)^2}\right)=\sum_{n=0}^{\infty}f_n(x)t^n, C\in\mathbb{C}^{r\times r}. \tag{5.2}$$

The polynomials $f_n(x)$ defined by (5.1) and (5.2) have the following properties [2]:

$$f_n(x)=\frac{(C)_n}{n!}\sum_{k=0}^{n}(-nI)_k(C+nI)_k\left[\left(\frac{C}{2}\right)_k\left(\frac{C}{2}+\frac{I}{2}\right)_k\right]^{-1}\gamma_k\ x^k,$$

$$xf_n'(x)-nf_n(x)=-(C+(n-1)\,I)f_{n-1}(x)-xf_{n-1}'(x)\ ,\ n\geq 1,$$

$$xf_n'(x)-nf_n(x)=-C\sum_{k=0}^{n-1}f_k(x)-2x\sum_{k=0}^{n-1}f_k'(x)\ ,\ n\geq 1,$$

$$xf_n'(x)-nf_n(x)=\sum_{k=0}^{n-1}(-1)^{n-k}(C+2kI)f_k(x)\ ,\ n\geq 1,$$

where $C\in\mathbb{C}^{r\times r}$ and $C+nI$ is invertible for every integer $n\geq 0$.

If we choose

$$C=A+B+I\quad ;\ \gamma_n=\frac{(I+A+B)_{2n}}{2^{2n}n!}(I+A)_n^{-1}$$

in Theorem 5.2, we see that the matrix polynomials f_n is

$f_n(v)=\{c\,(a-b)\}^{-n}\,(I+A+B)_nF_n^{(A,B)}\,((b-a)\,v{+}a\ ;a,b,c)\,(I+A)_n^{-1}.$

Hence, Theorem 5.2 gives following results, when put in terms of x rather than v.

Theorem 5.3. *The polynomials EJMPs have the following recurrence relations:*

$$\begin{aligned}(x-a)\Big[(A+B+nI)\,\frac{d}{dx}F_n^{(A,B)}\,(x;a,b,c)\\ +c\,(a-b)\,\frac{d}{dx}F_{n-1}^{(A,B)}\,(x;a,b,c)\,(A+nI)\Big]\\ =\ (A+B+nI)\Big[nF_n^{(A,B)}\,(x;a,b,c)\\ -c\,(a-b)\,F_{n-1}^{(A,B)}\,(x;a,b,c)\,(A+nI)\Big],\end{aligned}$$

$$(x-a)\frac{d}{dx}F_n^{(A,B)}(x;a,b,c)-nF_n^{(A,B)}(x;a,b,c)$$

$$= \; -\{c(a-b)\}^n(A+B+I)_n^{-1}\sum_{k=0}^{n-1}\{c(a-b)\}^{-k}(A+B+I)_k$$

$$.\left\{(A+B+I)F_k^{(A,B)}(x;a,b,c)+2(x-a)\frac{d}{dx}F_k^{(A,B)}(x;a,b,c)\right\}$$

$$.(I+A)_k^{-1}(I+A)_n$$

and

$$(x-a)\frac{d}{dx}F_n^{(A,B)}(x;a,b,c)-nF_n^{(A,B)}(x;a,b,c)$$

$$= \; \{c(b-a)\}^n(A+B+I)_n^{-1}\sum_{k=0}^{n-1}\{c(b-a)\}^{-k}(A+B+(2k+1)I)$$

$$.(A+B+I)_kF_k^{(A,B)}(x;a,b,c)(I+A)_k^{-1}(I+A)_n$$

where all eigenvalues z of the matrices A and B of the extended Jacobi matrix polynomials $F_n^{(A,B)}(x;a,b,c)$ satisfy the condition $Re(z)>-1$ and $A+B+nI$ is invertible for every integer $n\geq 0$.

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Rabia AKTAŞ
Ankara University, Faculty of Science,
Department of Mathematics,
Tandoğan TR-06100, Ankara, Turkey.
E-Mail address (R. Aktaş): raktas@science.ankara.edu.tr

Bayram ÇEKİM
Gazi University, Faculty of Sciences and Arts,
Department of Mathematics,
Teknikokullar TR-06500, Ankara, Turkey.
E-mail address (B. Çekim): bayramcekim@gazi.edu.tr

Ali ÇEVİK
Ankara University, Faculty of Science,
Department of Mathematics,
Tandoğan TR-06100, Ankara, Turkey.
E-Mail address (A. Çevik): cevik@science.ankara.edu.tr