Contemporary Mathematics
Volume **665**, 2016
http://dx.doi.org/10.1090/conm/665/13364

On some transcendental values of the p-adic gamma function

Hamza Menken and Abdulkadir Aşan

Abstract. We study on the p-adic gamma function. We give some results on transcendental values of the p-adic gamma function at certain p-adic Liouville arguments.

1. Introduction

The classical gamma function $\Gamma(z)$ is defined by the formula

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

for any complex number with $\Re(z) > 0$. The transcendence of the values of the gamma function $\Gamma(z)$ at rational arguments have been studied for many decades. For instance, $\Gamma(1/2)$ is transcendental and equals to $\sqrt{\pi}$. Also, the numbers $\Gamma(1/3)$ and $\Gamma(1/4)$ are transcendental ([**6**]). The algebraic independent of some values of the gamma function $\Gamma(z)$ at some rational arguments and certain transcendental numbers were proved in [**11**], [**20**], [**25**] (for detail see [**27**]).

The present paper is devoted to study of the p-adic analogue of the gamma function at certain p-adic integers values. Let p be a prime number and let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. For simplicity we assume that p is a odd prime number.

It is well known that the p-adic gamma function Γ_p is defined by Y. Morita (1975) [**19**] as the continuous extension to $\mathbb{Z}_p$ of the function $n \to (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} j$,

that is, $\Gamma_p(x)$ is defined by the formula

$$\Gamma_p(x) = \lim_{n \to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} j$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. The p-adic gamma function $\Gamma_p(x)$ had been studied by J. Diamond (1977) [**9**], D. Barsky (1979) [**2**], T. K. Kim (1997) [**13**]. and others. The relationship between some special functions and the p-adic gamma function $\Gamma_p(x)$ were investigated by B. Gross and N. Koblitz (1975) [**12**], H. Cohen and E. Friedman (2008) [**8**] and I. Shapiro (2012) [**24**].

2010 *Mathematics Subject Classification.* Primary 11J61, 11S80.
Key words and phrases. p-adic gamma function, p-adic number, p-adic Liouville number.

We recall some properties of p-adic gamma function Γ_p: (for details see [**10**] and [**23**])
(i) For all $x \in \mathbb{Z}_p$

$$\Gamma_p(x+1) = h_p(x)\Gamma_p(x) \tag{1.1}$$

where

$$h_p(x) := \begin{cases} -x \text{ if } |x|_p = 1 \\ -1 \text{ if } |x|_p < 1 \end{cases}$$

(ii) For all $x, y \in \mathbb{Z}_p$

$$|\Gamma_p(x) - \Gamma_p(y)|_p \leq |x-y|_p \,. \tag{1.2}$$

and

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\ell(x)} \qquad (x \in \mathbb{Z}_p)$$

where $\ell : \mathbb{Z}_p \to \{1, 2, ..., p\}$ assigns to $x \in \mathbb{Z}_p$ its residue $\in \{1, 2, ..., p\}$ modulo $p\mathbb{Z}_p$. Hence,

$$\Gamma_p(\frac{1}{2})^2 = (-1)^{\ell(\frac{1}{2})}$$

and

$$\Gamma_p(\frac{1}{2})^2 = \begin{cases} 1 \text{ if } p \equiv 3 \ (mod\ 4) \\ -1 \text{ if } p \equiv 1 \ (mod\ 4) \end{cases}$$

Thus, the role played by $\pi = 3.14...$ is taken over by one of the numbers $-1, 1$. This formula yields a proof for the existence of $\sqrt{-1}$ in $\mathbb{Q}_p$ in the case of $p \equiv 1$ $(mod\ 4)$. Also, if $p \equiv 3$ $(mod\ 4)$) then

$$\Gamma_p(\frac{1}{2}) = \begin{cases} 1 \text{ if } (\frac{1}{2}(p-1))! \equiv 1 \ (mod\ p) \\ -1 \text{ if } (\frac{1}{2}(p-1))! \equiv -1 \ (mod\ p) \end{cases}$$

and if $p \equiv 1$ $(mod\ 4)$ then $i \in \mathbb{Q}_p, i^2 = -1$ and

$$\Gamma_p(\frac{1}{2}) = \begin{cases} i \text{ if } (\frac{1}{2}(p-1))! \equiv -\bar{i} \ (mod\ p) \\ -i \text{ if } (\frac{1}{2}(p-1))! \equiv \bar{i} \ (mod\ p) \end{cases}$$

In contrast to real case, the number $\Gamma_p(\frac{1}{2})$ is algebraic. Also, it is clear that for all natural number n, by the property (1.1) the numbers $\Gamma_p(n + \frac{1}{2})$ are algebraic.

In the present paper we discuss the transcendence of the p-adic gamma function $\Gamma_p(\lambda)$ when λ is a p-adic Liouville number. Now, we recall the definition of a p-adic Liouville number.

Definition 1.1. ([**7**], [**23**]) *Let λ be a p-adic integer. If*

$$\lim_{n\to\infty} \inf \sqrt[n]{|n-\lambda|_p} = 0,$$

then λ is called a p-adic Liouville number.

According to this definition, $\lambda \in \mathbb{Z}_p$ is *a p-adic Liouville number if and only if there exists a sequence* of *positive integers a_n such that*

$$\lim_{n\to\infty} \sqrt[n]{|a_n - \lambda|_p} = 0.$$

It is clear that the series $\alpha = \sum_{n=0}^{\infty} p^{n!}$ converges for all prime p and its sum is a p-adic Liouville number.

The definition above is first introduced by D. Clark [**7**] and it is better adapted to p-adic differential equations, and also there are many applications to the p-adic differential equations.(for details see [**22**]). We note that the set of p-adic Liouville numbers forms a dense subset of $\mathbb{Z}_p$ and every p-adic Liouville number is transcendental over $\mathbb{Q}$ (for detail see [**23**]).

In general case the p-adic transcendental numbers have been studied by K. Mahler (1939) [**15**], W. W. Adams (1966) [**1**], X. Long Xin (1989) [**26**], K. Nishioka (1990) [**21**] and others. As a special case the p-adic Liouville numbers have been studied in [**3**], [**5**], [**16**]-[**18**] and others (For detail see [**4**], Section 9.3) .

In the present work we show that if $\lambda \in \mathbb{Z}_p$ is a p-adic Liouville number the $\Gamma_p(\lambda)$ must be transcendental. Also, to guarantee that $\Gamma_p(\lambda)$ is a p-adic Liouville number we give some conditions on $\lambda \in \mathbb{Z}_p$.

2. Main Results

Fist we obtain the following result.

LEMMA 2.1. *If* $\lambda \in \mathbb{Z}_p \backslash \mathbb{N}$, *then* $\Gamma_p(\lambda) \in \mathbb{Z}_p \backslash \mathbb{N}$.

PROOF. Let

$$\lambda = a_0 + a_1 p + \cdots + a_n p^n + \cdots$$

be the p-adic expansion of λ. Since $\lambda \in \mathbb{Z}_p \backslash \mathbb{N}$, there exist infinitely many natural numbers n such that $a_n \neq 0$. By a_{n_k} we denote the all non-zero terms a_n, means that $a_{n_k} \neq 0$ for all $k = 1, 2, \cdots$ and the other terms equal to zero. Let $s_n = a_0 + a_1 p + \cdots + a_n p^n$ be the partial sum of λ. Hence,

$$s_{n_k} = a_{n_0} p^{n_0} + a_{n_1} p^{n_1} + \cdots + a_{n_k} p^{n_k} = \sum_{i=0}^{k} a_{n_i} p^{n_i}.$$

Since $\lim_{k\to\infty} s_{n_k} = \lambda$ (with respect to the p-adic norm $|\cdot|_p$), we can write

$$\Gamma_p(\lambda) = \lim_{k\to\infty} \Gamma_p(s_{n_k}) = \lim_{k\to\infty} (-1)^{s_{n_k}} \prod_{\substack{1\le j < s_{n_k} \\ (p,j)=1}} j.$$

Now we must show that $\Gamma_p(\lambda) \in \mathbb{Z}_p \backslash \mathbb{N}$. Assume that $\Gamma_p(\lambda) \in \mathbb{N}$. Then, the p-adic expansion of $\Gamma_p(\lambda)$ must be finite. Let the p-adic expansion of $\Gamma_p(\lambda)$ be

$$\Gamma_p(\lambda) = b_0 + b_1 p + \cdots + b_m p^m.$$

where $0 \le b_i \le p-1$ and $b_m \neq 0$.
Since $\lim_{k\to\infty} \Gamma_p(s_{n_k}) = \Gamma_p(\lambda)$, the p-adic expansion of $\Gamma_p(s_{n_k})$ must be finite and equal to $\Gamma_p(\lambda) = b_0 + b_1 p + \cdots + b_m p^m$ for $k \ge k_0$. But, it is impossible. In fact, for any $k_2 > k_1 \ge k_0$, then we can write

$$\begin{aligned} s_{n_{k_1}} &= a_{n_0} p^{n_0} + a_{n_1} p^{n_1} + \cdots + a_{n_{k_1}} p^{n_{k_1}} \\ s_{n_{k_2}} &= a_{n_0} p^{n_0} + a_{n_1} p^{n_1} + \cdots + a_{n_{k_1}} p^{n_{k_1}} + \ldots + a_{n_{k_2}} p^{n_{k_2}}. \end{aligned}$$

Since $a_{n_k} \neq 0$ for all positive integer k we conclude that $\Gamma_p(s_{n_{k_1}}) \neq \Gamma_p(s_{n_{k_2}})$. Hence, the p-adic expansion of $\Gamma_p(\lambda)$ can not be finite, and $\Gamma_p(\lambda) \in \mathbb{Z}_p \backslash \mathbb{N}$. □

Now by using the lemma we prove the following results.

THEOREM 2.2. *If* $\lambda \in \mathbb{Z}_p$ *is a* p*-adic Liouville number, then* $\Gamma_p(\lambda)$ *is transcendental.*

Proof. Since $\lambda \in \mathbb{Z}_p$ is a p-adic Liouville number, then by Definition 1.1

$$\lim_{n\to\infty} \inf |\lambda - n|_p^{\frac{1}{n}} = 0.$$

So, for any $\varepsilon > 0$ there is a sequence of natural numbers (z_n) such that

$$0 < |\lambda - z_n|_p^{\frac{1}{n}} < \varepsilon$$

for all $n \in \mathbb{Z}^+$. It is clearly that $\Gamma_p(z_n) \in \mathbb{Z}$, since $(z_n) \subseteq \mathbb{N}$. By the property (1.2) we can write

$$|\Gamma_p(\lambda) - \Gamma_p(z_n)|_p^{\frac{1}{n}} \le |\lambda - z_n|_p^{\frac{1}{n}} < \varepsilon.$$

Now we show that $\Gamma_p(\lambda)$ is transcendental. Assume that $\Gamma_p(\lambda)$ is an algebraic number of degree m (≥ 1). By the p-adic version of Thue-Siegel theorem ([**14**]), the inequality

$$0 < \left|\Gamma_p(\lambda) - \frac{a}{b}\right|_p < \frac{1}{H\left(\frac{a}{b}\right)^m}$$

has only finitely many solutions in $\mathbb{Q}$. We denote these solutions by $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_N}{b_N}$. Let $M = \max_{1\le i\le N} \left\{H\left(\frac{a_i}{b_i}\right)\right\}$ where $H\left(\frac{a_i}{b_i}\right)$ is the height of the number $\frac{a_i}{b_i}$, i.e., $H\left(\frac{a_i}{b_i}\right) = \max\{|a_i|, |b_i|\}$. Hence, the inequality

$$0 < |\Gamma_p(\lambda) - x|_p < \frac{1}{M^m}$$

has only finitely many solutions in $\mathbb{Q}$. But, if we choose $\varepsilon = \frac{1}{M}$, since $\lambda \in \mathbb{Z}_p$ is a p-adic Liouville number, there is a natural number N_0 such that the inequality

$$0 < |\Gamma_p(\lambda) - \Gamma_p(z_n)|_p \le |\lambda - z_n|_p < \frac{1}{M^n}.$$

holds for all $n \ge N_0$. By Lemma 2.1, $\Gamma_p(\lambda)$ has an infinitely p-adic expansions and the sequence $\Gamma_p(z_n)$ have infinitely many distinct values. Thus, the inequality

$$0 < |\Gamma_p(\lambda) - \Gamma_p(z_n)|_p < \frac{1}{M^n} \le \frac{1}{M^m}$$

holds for all $n \ge \max\{N_0, m\}$. This contradiction shows that $\Gamma_p(\lambda)$ is transcendental. □

Theorem 2.3. *Let $\lambda \in \mathbb{Z}_p$. Suppose for each $\varepsilon > 0$ there exists a sequence of even natural number (z_n) such that*

$$0 < |\lambda - z_n|_p^{\frac{1}{n}} < \varepsilon.$$

for all $n \ge n_0$. Then,
a) λ is a p-adic Liouville number,
b) $\Gamma_p(\lambda)$ is a p-adic Liouville number.

Proof. a) It is clear from the definition of a Liouville number.

b)According to Lemma 2.1, we have $\Gamma_p(\lambda) \in \mathbb{Z}_p \backslash \mathbb{N}$. We know that the inequality

$$|\Gamma_p(x) - \Gamma_p(y)|_p \le |x - y|_p$$

holds for all $x, y \in \mathbb{Z}_p$. For given any $\varepsilon > 0$, using this inequality and the hypothesis of the theorem, there exists $n_0 \in \mathbb{N}$ such that

$$|\Gamma_p(\lambda) - \Gamma_p(z_n)|_p^{\frac{1}{n}} \le |\lambda - z_n|_p^{\frac{1}{n}} < \varepsilon$$

for all $n \geq n_0$. Hence, the inequality

$$|\Gamma_p(\lambda) - \Gamma_p(z_n)|_p^{\frac{1}{n}} < \varepsilon$$

holds for all $n \geq n_0$ Since (z_n) a sequence of even natural numbers, we have that $\Gamma_p(z_n) \in \mathbb{N}$. On the other hand, it follows from the conditions

$$\Gamma_p(\lambda) \in \mathbb{Z}_p \backslash \mathbb{N}$$

and

$$\lim_{n\to\infty} \Gamma_p(z_n) = \Gamma_p(\lambda)$$

that $(\Gamma_p(z_n))$ is an infinitely sequence of natural numbers Thus, we have

$$\lim_{n\to\infty} \inf |\Gamma_p(\lambda) - \Gamma_p(z_n)|_p^{\frac{1}{n}} = 0.$$

By Definition 1.1, $\Gamma_p(\lambda)$ is a p-adic Liouville number. □

Acknowledgement:

This work is supported by Mersin University. The authors would like to thank the reviewers for their useful suggestions.

References

[1] William W. Adams, *Transcendental numbers in the P-adic domain*, Amer. J. Math. **88** (1966), 279–308. MR0197399 (33 #5564)

[2] Daniel Barsky, *On Morita's p-adic gamma function*, Math. Proc. Cambridge Philos. Soc. **89** (1981), no. 1, 23–27, DOI 10.1017/S030500410005790X. MR591968 (82a:12011)

[3] V. V. Beresnevich, V. I. Bernik, and E. I. Kovalevskaya, *On approximation of p-adic numbers by p-adic algebraic numbers*, J. Number Theory **111** (2005), no. 1, 33–56, DOI 10.1016/j.jnt.2004.09.007. MR2124042 (2006e:11097)

[4] Yann Bugeaud, *Approximation by algebraic numbers*, Cambridge Tracts in Mathematics, vol. 160, Cambridge University Press, Cambridge, 2004. MR2136100 (2006d:11085)

[5] Tuangrat Chaichana, Takao Komatsu, and Vichian Laohakosol, *Liouville numbers in the non-Archimedean case*, Publ. Math. Debrecen **77** (2010), no. 1-2, 39–63. MR2675733 (2011g:11131)

[6] G. V. Čudnovsʹkiĭ, *Algebraic independence of constants connected with the exponential and the elliptic functions* (Russian, with English summary), Dokl. Akad. Nauk Ukrain. SSR Ser. A **8** (1976), 698–701, 767. MR0424711 (54 #12670)

[7] D. N. Clark, *A note on the p-adic convergence of solutions of linear differential equations*, Proc. Amer. Math. Soc. **17** (1966), 262–269. MR0186895 (32 #4350)

[8] Henri Cohen and Eduardo Friedman, *Raabe's formula for p-adic gamma and zeta functions* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **58** (2008), no. 1, 363–376. MR2401225 (2009d:11165)

[9] Jack Diamond, *The p-adic log gamma function and p-adic Euler constants*, Trans. Amer. Math. Soc. **233** (1977), 321–337. MR0498503 (58 #16610)

[10] Bernard Dwork, *A note on the p-adic gamma function*, Study group on ultrametric analysis, 9th year: 1981/82, No. 3 (Marseille, 1982), Inst. Henri Poincaré, Paris, 1983, pp. Exp. No. J5, 10. MR720576 (85j:11171)

[11] Pierre Grinspan, *Measures of simultaneous approximation for quasi-periods of abelian varieties*, J. Number Theory **94** (2002), no. 1, 136–176, DOI 10.1006/jnth.2001.2733. MR1904966 (2003k:11122)

[12] Benedict H. Gross and Neal Koblitz, *Gauss sums and the p-adic Γ-function*, Ann. of Math. (2) **109** (1979), no. 3, 569–581, DOI 10.2307/1971226. MR534763 (80g:12015)

[13] T. K. Kim, A note on analogue of gamma functions, in Proc. Confer. on 5th Trancendental Number Theory 5, No.1, 111-118, Gakushin Univ., Tokyo, Japan, 1997.

[14] Kurt Mahler, *Zur Approximation algebraischer Zahlen. I* (German), Math. Ann. **107** (1933), no. 1, 691–730, DOI 10.1007/BF01448915. MR1512822

[15] K. Mahler, Uber eine Klassen-Einteilung der p-adischen Zahlen, Mathematica B, Tijdschr. V. Stud. 3, 177-185 (1935).

[16] Hamza Menken, *An investigation on p-adic U numbers*, İstanbul Üniv. Fen Fak. Mat. Derg. **59** (2000), 111–143 (2001). MR1882012 (2002k:11116)

[17] Hamza Menken and Khanlar R. Mamedov, *Point on curves whose coordinates are p-adic U-numbers*, p-adic mathematical physics, AIP Conf. Proc., vol. 826, Amer. Inst. Phys., Melville, NY, 2006, pp. 267–273, DOI 10.1063/1.2193128. MR2258691 (2007m:11094)

[18] Hamza Menken and Abdulkadir Aşan, *On some properties of Liouville numbers in the non-Archimedean case*, Eur. J. Pure Appl. Math. **6** (2013), no. 2, 239–246. MR3053445

[19] Yasuo Morita, *A p-adic analogue of the Γ-function*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), no. 2, 255–266. MR0424762 (54 #12720)

[20] Yu. V. Nesterenko, *Modular functions and transcendence questions* (Russian, with Russian summary), Mat. Sb. **187** (1996), no. 9, 65–96, DOI 10.1070/SM1996v187n09ABEH000158; English transl., Sb. Math. **187** (1996), no. 9, 1319–1348. MR1422383 (97m:11102)

[21] Kumiko Nishioka, *p-adic transcendental numbers*, Proc. Amer. Math. Soc. **108** (1990), no. 1, 39–41, DOI 10.2307/2047690. MR994783 (90e:11099)

[22] Marius van der Put and Lenny Taelman, *Local p-adic differential equations*, p-adic mathematical physics, AIP Conf. Proc., vol. 826, Amer. Inst. Phys., Melville, NY, 2006, pp. 291–297, DOI 10.1063/1.2193131. MR2258694 (2007i:12007)

[23] W. H. Schikhof, *Ultrametric calculus*, Cambridge Studies in Advanced Mathematics, vol. 4, Cambridge University Press, Cambridge, 2006. An introduction to p-adic analysis; Reprint of the 1984 original [MR0791759]. MR2444734

[24] Ilya Shapiro, *Frobenius map and the p-adic gamma function*, J. Number Theory **132** (2012), no. 8, 1770–1779, DOI 10.1016/j.jnt.2012.03.005. MR2922344

[25] K. G. Vasil′ev, *On the algebraic independence of the periods of Abelian integrals* (Russian, with Russian summary), Mat. Zametki **60** (1996), no. 5, 681–691, 799, DOI 10.1007/BF02309165; English transl., Math. Notes **60** (1996), no. 5-6, 510–518 (1997). MR1619893 (99c:11094)

[26] Xiao Long Xin, *Mahler's classification of p-adic numbers* (Chinese, with English summary), Pure Appl. Math. (Xi'an) **5** (1989), 73–80. MR1304462 (95h:11066)

[27] Michel Waldschmidt, *Transcendence of periods: the state of the art*, Pure Appl. Math. Q. **2** (2006), no. 2, 435–463, DOI 10.4310/PAMQ.2006.v2.n2.a3. MR2251476 (2007d:11083)

Mersin University, Science and Arts Faculty, Mathematics Department, 33343 Ciftlikkoy Campus, Mersin-Turkey

E-mail address: hmenken@mersin.edu.tr

Mersin University, Institute of Science, Mathematics Graduate Program, 33343 Ciftlikkoy Campus, Mersin-Turkey

E-mail address: akadirasan@mersin.edu.tr