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ON THE (s,t)-PADOVAN AND (s,t)-PERRIN QUATERNIONS

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Abstract. In this paper we first introduce a class of (s,t)-Padovan and (s,t)-Perrin quaternions which generalizes Padovan and Perrin quaternions, and then we derive new Binet-like formulas, generating functions and certain binomial sums for these quaternions.

1. INTRODUCTION

Special number sequences have play important role in mathematics and applied sciences. Moreover, some special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan and Perrin sequences have many applications in art, music, photography, architecture, painting, engineering, geometry and others. It is well-known that the term golden ratio is defined the ratio of two consecutive Fibonacci numbers converges to

$$\frac{1+\sqrt{5}}{2} \approx 1.618034.$$

The golden ratio has many applications in engineering, physics, architecture, arts and other. In similar way, the ratio of two consecutive Padovan or Perrin numbers converges to

$$\sqrt[3]{\frac{1}{2}+\frac{1}{6}\sqrt{\frac{23}{3}}}+\sqrt[3]{\frac{1}{2}-\frac{1}{6}\sqrt{\frac{23}{3}}} \approx 1.324718,$$

that is called as "plastic ratio". The plastic ratio (number) was first defined by Gerard Cordonnier in 1924. He described applications to architecture and illustrated the use of the plastic number in many buildings. Furthermore, the plastic number is the unique real root of the equation $x^3 - x - 1 = 0$, the characteristic equation of Padovan number sequences (see [5, 7, 13]).

The Padovan sequence $\{P_n\}_{n\geq 0}$ is defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_{n+3} = P_{n+1} + P_n, \quad \text{for all } n \geq 0. \tag{1.1}$$

First few terms of this sequence are 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28. Designated by I. Stewart as the Padovan numbers in honour of the contemporary architect Richard Padovan, they apparently have a more extensive lineage [10].

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The Perrin sequence $\{R_n\}_{n\geq 0}$ is defined by the initial values $R_0 = 3$, $R_1 = 0$ and $R_2 = 2$ and the recurrence relation

$$R_{n+3} = R_{n+1} + R_n, \quad \text{for all } n \geq 0. \tag{1.2}$$

First few terms of Perrin sequence are 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29. This sequence was first discussed by Edouard Lucas in 1878 [8]. Although the study of Perrin numbers started in the beginning of nineteenth century under the different names, the master study was done by Shannon et al in 2006 [9]. Other investigations regarded to Padovan and Perrin sequences can be found in [2, 7, 12, 13].

A generalization of the Padovan sequence $\{P_n\}_{n\geq 0}$, which is called the (s,t)-Padovan sequence, say $\{\mathcal{P}_n(s,t)\}_{n\geq 0}$ is defined by the following recurrence relation for $n \geq 0$ and any integer numbers $s > 0$ and $t \neq 0$ such that $27t^2 - 4s^3 \neq 0$:

$$\mathcal{P}_{n+3}(s,t) = s\mathcal{P}_{n+1}(s,t) + t\mathcal{P}_n(s,t),$$

where $\mathcal{P}_0(s,t) = 0$, $\mathcal{P}_1(s,t) = 1$ and $\mathcal{P}_2(s,t) = 0$. To simplify notation, we take $\mathcal{P}_n(s,t) = \mathcal{P}_n$.

A generalization of the Perrin sequence $\{R_n\}_{n\geq 0}$, which is called the (s,t)-Perrin sequence, say $\{\mathcal{R}_n(s,t)\}_{n\geq 0}$ is defined by the following recurrence relation for $n \geq 0$ and any integer numbers $s > 0$ and $t \neq 0$ such that $27t^2 - 4s^3 \neq 0$:

$$\mathcal{R}_{n+3}(s,t) = s\mathcal{R}_{n+1}(s,t) + t\mathcal{R}_n(s,t),$$

where $\mathcal{R}_0(s,t) = 3$, $\mathcal{R}_1(s,t) = 0$ and $\mathcal{R}_2(s,t) = 2s$. To simplify notation, we take $\mathcal{R}_n(s,t) = \mathcal{R}_n$. The (s,t)-Padovan and (s,t)-Perrin sequences were investigated in [2].

For every $x \in \mathbb{N}$, one can write the Binet-like formulas for the (s,t)-Padovan and (s,t)-Perrin sequences as the form

$$\mathcal{P}_n = a\alpha^n + b\beta^n + c\gamma^n \quad \text{and} \quad \mathcal{R}_n = \alpha^n + \beta^n + \gamma^n$$

where α, β and γ are the roots of the characteristic equation

$$x^3 - sx - t = 0 \tag{1.3}$$

associated with (1.1) and (1.2), where

$$a = \frac{(\beta-1)(\gamma-1)}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{(\alpha-1)(\gamma-1)}{(\beta-\alpha)(\beta-\gamma)}, \quad c = \frac{(\alpha-1)(\beta-1)}{(\alpha-\gamma)(\beta-\gamma)}.$$

The Binet-like formulas for the (s,t)-Padovan and (s,t)-Perrin sequences were given in [2].

The quaternions were first described by Irish mathematician William Rowan Hamilton in 1843 as a number system that extends the complex numbers. The quaternions play an important role in diverse areas such that as kinematics [1], chemistry [3], mechanics [4] and quantum mechanics [11]. A quaternion is defined by

$$q = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$$

where a_0, a_1, a_2 and a_3 are real numbers and $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$ are the standard basis in $\mathbb{R}^4$. The quaternion multiplication is defined using the rules:

$$e_0^2 = 1, \quad e_1^2 = e_2^2 = e_3^2 = -1,$$
$$e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1 \quad \text{and} \quad e_3e_1 = -e_1e_3 = e_2.$$

This algebra is associative and non-commutative. Let $q = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3$ and $p = b_0e_0 + b_1e_1 + b_2e_2 + b_3e_3$ be any two quaternions. Then the addition and subtraction of them is

$$q \mp p = (a_0 \mp b_0)e_0 + (a_1 \mp b_1)e_1 + (a_2 \mp b_2)e_2 + (a_3 \mp b_3)e_3$$

and for $k \in \mathbb{R}$, the multiplication by scalar is

$$kq = ka_0e_0 + ka_1e_1 + ka_2e_2 + ka_3e_3$$

and the conjugate and norm of a quaternion are, respectively,

$$\overline{q} = a_0e_0 - a_1e_1 - a_2e_2 - a_3e_3$$

and addition, equality and multiplication by scalar of two quaternions

$$N(q) = q\overline{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

can be found in [2, 5, 6, 12].

The nth Padovan quaternions was described by Tasci in [12] as

$$\mathcal{Q}P_n = P_ne_0 + P_{n+1}e_1 + P_{n+2}e_2 + P_{n+3}e_3,$$

where P_n is nth Padovan number.

Tasci [12] gave Binet-like formula and a generating function for the Padovan quaterinions.

In the present work we define the nth Perrin quaternions by the formula

$$\mathcal{Q}R_n = R_ne_0 + R_{n+1}e_1 + R_{n+2}e_2 + R_{n+3}e_3,$$

where R_n is the nth Perrin number.

2. MAIN RESULTS

In this section, we define two new quaternions that are (s,t)-Padovan and (s,t)-Perrin quaternions. Then, we give their Binet-like formulas, generating functions and some certain binomial sum formulas.

Definition 2.1. The (s,t)-Padovan quaternion sequence $\{\mathcal{QP}_n(s,t)\}_{n\geq 0}$ is defined by

$$\mathcal{QP}_n(s,t) = \mathcal{P}_ne_0 + \mathcal{P}_{n+1}e_1 + \mathcal{P}_{n+2}e_2 + \mathcal{P}_{n+3}e_3, \tag{2.1}$$

where $\mathcal{P}_n$ is the nth (s,t)-Padovan number. To simplify notation, we take $\mathcal{QP}_n(s,t) = \mathcal{QP}_n$.

Definition 2.2. The (s,t)-Perrin quaternion sequence $\{\mathcal{QR}_n(s,t)\}_{n\geq 0}$ is defined by

$$\mathcal{QR}_n(s,t) = \mathcal{R}_ne_0 + \mathcal{R}_{n+1}e_1 + \mathcal{R}_{n+2}e_2 + \mathcal{R}_{n+3}e_3, \tag{2.2}$$

where $\mathcal{R}_n$ is the nth (s,t)-Perrin number. To simplify notation, we take $\mathcal{QR}_n(s,t) = \mathcal{QR}_n$.

Theorem 2.1 (Binet-like formula)**.** *The Binet-like formulas for the nth (s,t)-Padovan quaternion is*

$$\mathcal{QP}_n = a\underline{\alpha}\alpha^n + b\underline{\beta}\beta^n + c\underline{\gamma}\gamma^n, \quad n \geqslant 0, \tag{2.3}$$

where

$$\underline{\alpha} = e_0 + \alpha e_1 + \alpha^2e_2 + \alpha^3e_3, \quad \underline{\beta} = e_0 + \beta e_1 + \beta^2e_2 + \beta^3e_3, \quad \underline{\gamma} = e_0 + \gamma e_1 + \gamma^2e_2 + \gamma^3e_3.$$

Proof. From the definition of n^{th} (s,t)-Padovan quaternion $\mathcal{QP}_n$ in (2.1) and Binet-like formula for the nth (s,t)-Padovan number $\mathcal{P}_n$, we write

$$\begin{aligned}\mathcal{QP}_n &= \mathcal{P}_ne_0 + \mathcal{P}_{n+1}e_1 + \mathcal{P}_{n+2}e_2 + \mathcal{P}_{n+3}e_3\\ &= (a\alpha^n + b\beta^n + c\gamma^n)e_0 + (a\alpha^{n+1} + b\beta^{n+1} + c\gamma^{n+1})e_1 + (a\alpha^{n+2}\\ &\quad + b\beta^{n+2} + c\gamma^{n+2})e_2 + (a\alpha^{n+3} + b\beta^{n+3} + c\gamma^{n+3})e_3\\ &= a(e_0 + \alpha e_1 + \alpha^2e_2 + \alpha^3e_3)\alpha^n + b(e_0 + \beta e_1 + \beta^2e_2 + \beta^3e_3)\beta^n\\ &\quad + c(e_0 + \gamma e_1 + \gamma^2e_2 + \gamma^3e_3)\gamma^n = a\underline{\alpha}\alpha^n + b\underline{\beta}\beta^n + c\underline{\gamma}\gamma^n.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.2 (Binet-like formula)**.** *The Binet-like formula for the nth (s,t)-Perrin quaternion is*

$$\mathcal{QR}_n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n + \underline{\gamma}\gamma^n, \quad n \geqslant 0, \tag{2.4}$$

where

$$\underline{\alpha} = e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3, \quad \underline{\beta} = e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3, \quad \underline{\gamma} = e_0 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3.$$

Proof. From the definition of nth (s,t)-Perrin quaternion $\mathcal{QR}_n$ in (2.2) and Binet-like formula for the nth (s,t)-Perrin number $\mathcal{R}_n$, we write

$$\begin{aligned}\mathcal{QR}_n &= \mathcal{R}_n e_0 + \mathcal{R}_{n+1} e_1 + \mathcal{R}_{n+2} e_2 + \mathcal{R}_{n+3} e_3 \\ &= (\alpha^n + \beta^n + \gamma^n) e_0 + (\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1}) e_1 + (\alpha^{n+2} + \beta^{n+2} + \gamma^{n+2}) e_2 \\ &\quad + (\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3}) e_3 \\ &= (e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3)\alpha^n + (e_0 + \beta e_1 + \beta^2 e_2 + \beta^3 e_3)\beta^n \\ &\quad + (e_0 + \gamma e_1 + \gamma^2 e_2 + \gamma^3 e_3)\gamma^n = \underline{\alpha}\alpha^n + \underline{\beta}\beta^n + \underline{\gamma}\gamma^n.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.3. *The generating function for the nth (s,t)-Padovan quaternion is*

$$\mathcal{G}_{\mathcal{QP}}(x) = \frac{e_1 + s e_3 + (e_0 + s e_2 + t e_3)x + (t e_2)x^2}{1 - s x^2 - t x^3}.$$

Proof. Assume that the function

$$\mathcal{G}_{\mathcal{QP}}(x) = \sum_{n=0}^{\infty} \mathcal{QP}_n x^n = \mathcal{QP}_0 + \mathcal{QP}_1 x + \mathcal{QP}_2 x^2 + \mathcal{QP}_3 x^3 + \cdots + \mathcal{QP}_n x^n + \cdots$$

be generating function of the (s,t)-Padovan quaternions. Multiply both of side of the equality by the term $-sx^2$ such as

$$-sx^2 \mathcal{G}_{\mathcal{QP}}(x) = -s\mathcal{QP}_0 x^2 - s\mathcal{QP}_1 x^3 - s\mathcal{QP}_2 x^4 - s\ \mathcal{QP}_3 x^5 - \cdots - s\mathcal{QP}_n x^{n+2} - \cdots$$

and multiply by the term $-tx^3$ such as

$$-tx^3 \mathcal{G}_{\mathcal{QP}}(x) = -t\mathcal{QP}_0 x^3 - t\mathcal{QP}_1 x^4 - t\mathcal{QP}_2 x^5 - t\mathcal{QP}_3 x^6 - \cdots - t\mathcal{QP}_n x^{n+3} - \cdots$$

Then, we write

$$\begin{aligned}(1 - sx^2 - tx^3)\mathcal{G}_{\mathcal{QP}}(x) &= \mathcal{QP}_0 + \mathcal{QP}_1 x + (\mathcal{QP}_2 - s\mathcal{QP}_0)x^2 + (\mathcal{QP}_3 - s\mathcal{QP}_1 - t\mathcal{QP}_0)x^3 \\ &\quad + \cdots + (\mathcal{QP}_n - s\mathcal{QP}_{n-2} - t\mathcal{QP}_{n-3})x^n + \cdots\end{aligned}$$

Now, by using

$$\begin{gathered}\mathcal{QP}_0 = e_1 + s e_3, \\ \mathcal{QP}_1 = e_0 + s e_2 + t e_3, \\ \mathcal{QP}_2 = s e_1 + t e_2 + s^2 e_3, \\ \vdots \\ \mathcal{QP}_n - s\mathcal{QP}_{n-2} - t\mathcal{QP}_{n-3} = 0,\end{gathered}$$

we obtain that

$$\mathcal{G}_{\mathcal{QP}}(x) = \frac{e_1 + s e_3 + (e_0 + s e_2 + t e_3)x + (t e_2)x^2}{1 - s x^2 - t x^3}.$$

Thus, the proof is completed. □

Theorem 2.4. *The generating function of the nth (s,t)-Perrin quaternion is*
$$\mathcal{G}_{\mathcal{QR}}(x)=\frac{3e_0+2se_2+3te_3+(2e_1+3te_2+2s^2e_3)x+(-se_0+3te_1+2ste_3)x^2}{1-sx^2-tx^3}.$$

Proof. Let $\mathcal{G}_{\mathcal{QR}}(x)=\sum_{n=0}^{\infty}\mathcal{QR}_nx^n=\mathcal{QR}_0+\mathcal{QR}_1x+\mathcal{QR}_2x^2+\mathcal{QR}_3x^3+\cdots+\mathcal{QR}_nx^n+\cdots$ be generating function of the (s,t)-Perrin quaternions. Now multiply both of side of the equality by term $-sx^2$ such as
$$-sx^2\mathcal{G}_{\mathcal{QR}}(x)=-s\mathcal{QR}_0x^2-s\mathcal{QR}_1x^3-s\mathcal{QR}_2x^4-s\mathcal{QR}_3x^5-\cdots-s\mathcal{QR}_nx^{n+2}-\cdots$$
and multiply by $-tx^3$ such as
$$-tx^3\mathcal{G}_{\mathcal{QR}}(x)=-t\mathcal{QR}_0x^3-t\mathcal{QR}_1x^4-t\mathcal{QR}_2x^5-t\mathcal{QR}_3x^6-\cdots-t\mathcal{QR}_nx^{n+3}-\cdots.$$
Then, we write
$$\begin{aligned}(1-sx^2-tx^3)\mathcal{G}_{\mathcal{QR}}(x)&=\mathcal{QR}_0+\mathcal{QR}_1x+(\mathcal{QR}_2-s\mathcal{Q}\ \mathcal{R}_0)x^2+(\mathcal{QR}_3-s\mathcal{QR}_1-t\mathcal{QR}_0)x^3\\&\quad+\cdots+(\mathcal{QR}_n-s\mathcal{QR}_{n-2}-t\mathcal{QR}_{n-3})x^n+\cdots\end{aligned}$$
By using
$$\begin{aligned}\mathcal{QR}_0&=3e_0+2se_2+3te_3,\\ \mathcal{QR}_1&=2se_1+3te_2+2s^2e_3,\\ \mathcal{QR}_2&=2se_0+3te_1+2s^2e_2+5ste_3,\\ &\vdots\\ \mathcal{QR}_n-s\mathcal{QR}_{n-2}-t\mathcal{QR}_{n-3}&=0,\end{aligned}$$
we obtain that
$$\mathcal{G}_{\mathcal{QR}}(x)=\frac{3e_0+2se_2+3te_3+(2e_1+3te_2+2s^2e_3)x+(-se_0+3te_1+2ste_3)x^2}{1-sx^2-tx^3}.$$
This completes the proof. □

Theorem 2.5. *Let m be a positive integer. Then,*
$$\sum_{n=0}^{m}\binom{m}{n}s^nt^{m-n}\mathcal{QP}_n=\mathcal{QP}_{3m}.$$

Proof. Applying Binet-like formula (2.3), we obtain the identities
$$\begin{aligned}\sum_{n=0}^{m}\binom{m}{n}s^nt^{m-n}\mathcal{QP}_n&=\sum_{n=0}^{m}\binom{m}{n}s^nt^{m-n}(a\underline{\alpha}\alpha^n+b\underline{\beta}\beta^n+c\underline{\gamma}\gamma^n)\\&=\sum_{n=0}^{m}\binom{m}{n}(a\underline{\alpha}(s\alpha)^nt^{m-n}+b\underline{\beta}(s\beta)^nt^{m-n}+c\underline{\gamma}\ (s\gamma)^nt^{m-n}).\end{aligned}$$
Note that, for any any real numbers a and b, and any positive integer m, the identity
$$(a+b)^m=\sum_{n=0}^{m}\binom{m}{n}a^nb^{m-n}\tag{2.5}$$
holds. Hence $a\underline{\alpha}(s\alpha+t)^m+b\underline{\beta}(s\beta+t)^m+c\underline{\gamma}(s\gamma+t)^m$, $\alpha^3=s\alpha+t$, $\beta^3=s\beta+t$ and $\gamma^3=s\gamma+t$ are due to (1.3). Hence, $a\underline{\alpha}\alpha^{3m}+b\underline{\beta}\beta^{3m}+c\underline{\gamma}\gamma^{3m}$. Thus, the proof is completed. □

Theorem 2.6. *Let m be a positive integer. Then,*

$$\sum_{n=0}^{m}\binom{m}{n}s^{n}t^{m-n}\mathcal{QR}_{n}=\mathcal{QR}_{3m}.$$

Proof. Applying Binet-like formula (2.4) and combining this with (2.5) and (1.3) we obtain the identity

$$\begin{aligned}\sum_{n=0}^{m}\binom{m}{n}s^{n}t^{m-n}\mathcal{QR}_{n}&=\sum_{n=0}^{m}\binom{m}{n}s^{n}t^{m-n}(\underline{\alpha}\alpha^{n}+\underline{\beta}\beta^{n}+\underline{\gamma}\gamma^{n})\\&=\sum_{n=0}^{m}\binom{m}{n}(\underline{\alpha}\ (s\alpha)^{n}t^{m-n}+\underline{\beta}(s\beta)^{n}t^{m-n}+\underline{\gamma}(s\gamma)^{n}t^{m-n})\\&=\underline{\alpha}(s\alpha+t)^{m}+\underline{\beta}(s\beta+t)^{m}+\underline{\gamma}(s\gamma+t)^{m}=\underline{\alpha}\alpha^{3m}+\underline{\beta}\beta^{3m}+\underline{\gamma}\gamma^{3m}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.7. *Let m be a positive integer. Then,*

$$\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^{k}\mathcal{QP}_{n-k}=\mathcal{QP}_{n+2m}.$$

Proof. Applying Binet-like formula (2.3) and combining this with (2.5) and (1.3) we obtain the identity

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^{k}\mathcal{QP}_{n-k}&=\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^{k}(a\underline{\alpha}\alpha^{n-k}+b\underline{\beta}\beta^{n-k}+c\underline{\gamma}\gamma^{n-k})\\&=\sum_{k=0}^{m}\binom{m}{k}(a\underline{\alpha}\ (s\alpha)^{m-k}t^{k}\alpha^{n-m}+b\underline{\beta}(s\beta)^{m-k}t^{k}\beta^{n-m}\\&\quad+c\underline{\gamma}(s\gamma)^{m-k}t^{k}\gamma^{n-m})\\&=a\underline{\alpha}(s\alpha+t)^{m}\alpha^{n-m}+b\underline{\beta}\ (s\beta+t)^{m}\beta^{n-m}+c\underline{\gamma}(s\gamma+t)^{m}\gamma^{n-m}\\&=a\underline{\alpha}\alpha^{n+2m}+b\underline{\beta}\beta^{n+2m}+c\underline{\gamma}\gamma^{n+2m}.\end{aligned}$$

Thus, the proof is completed. □

Theorem 2.8. *Let m be a positive integer. Then,*

$$\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^{k}\mathcal{QR}_{n-k}=\mathcal{QR}_{n+2m}.$$

Proof. Applying Binet-like formula (2.4) and combining this with (2.5) and (1.3) we obtain the identity

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^{k}\mathcal{QR}_{n-k}&=\sum_{k=0}^{m}\binom{m}{k}s^{m-k}t^{k}(\underline{\alpha}\alpha^{n-k}+\underline{\beta}\beta^{n-k}+\underline{\gamma}\gamma^{n-k})\\&=\sum_{k=0}^{m}\binom{m}{k}\big(\underline{\alpha}\ (s\alpha)^{m-k}t^{k}\alpha^{n-m}+\underline{\beta}(s\beta)^{m-k}t^{k}\beta^{n-m}\\&\quad+\underline{\gamma}(s\gamma)^{m-k}t^{k}\gamma^{n-m}\big)\\&=\underline{\alpha}(s\alpha+t)^{m}\alpha^{n-m}+\underline{\beta}\ (s\beta+t)^{m}\beta^{n-m}+\underline{\gamma}(s\gamma+t)^{m}\gamma^{n-m}\end{aligned}$$

$$= \underline{\alpha}\alpha^{n+2m} + \underline{\beta}\beta^{n+2m} + \underline{\gamma}\gamma^{n+2m}.$$

Thus, the proof is completed. □

CONCLUSIONS

In this work, we define the (s,t)-Padovan and (s,t)-Perrin quaternions which generalize the Padovan and Perrin quaternions. By using some properties of the Padovan and Perrin quaternions, we derive new Binet-like formulas, generating functions for the (s,t)-Padovan and (s,t)-Perrin quaternions. Also, we give several binomial sum formulas related to these quaternions.

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