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DEFERRED CESÀRO MEAN AND DEFERRED STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

İLHAN DAĞADUR, ŞEYDA SEZGEK

Abstract. In this paper, the concepts of deferred Cesàro mean and deferred statistical convergence of double sequences are defined and some important results are obtained. Particularly in Section 3, we consider the case $\beta(n) = \lambda(n) - \lambda(n-1)$, $\gamma(m) = \mu(m) - \mu(m-1)$ for deferred Cesàro mean $D_{\beta,\gamma}$ where $\lambda = (\lambda(n))$ and $\mu = (\mu(m))$ are strictly increasing sequences of positive integers with $\lambda(0) = 0$ and $\mu(0) = 0$. Finally, some inclusion results between Cesàro submethod $C_{\lambda,\mu}$ and deferred Cesàro mean $D_{\beta,\gamma}$ for double sequences are obtained.

1. Introduction

The concept of statistical convergence was first introduced by Fast [6] and also independently by Buck [5] and Schoenberg [21] for real or complex sequences. Further, this concept was studied by Šalát[20], Fridy [7], Küçükaslan [11] and many others. Some equivalence results for Cesàro submethods have been studied by Goffman and Petersen [8], Armitage and Maddox [2] and Osikiewicz [16].

In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of the sequence $x = (x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive natural numbers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty.$$

In [4], the first study on double sequences was examined by Bromwich. And than it was investigated by many authors such as Hardy [10], Moricz [12], Tripathy [24], Başarır and Sonalcan [3]. The notion of regular convergence for double sequences was defined by Hardy [10]. After that both the theory of topological double sequence spaces and the theory of summability of double sequences were studied by Zeltser [25]. The statistical and Cauchy convergence for double sequences were examined by Mursaleen and Edely [13] and Tripathy [23] in recent years. Many recent

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improvements containing the summability by four dimensional matrices might be found in [17].

As a continuation of these works, we investigate deferred Cesàro mean and deferred statistical convergence for double sequences by using deferred double natural density of the subset of natural numbers and give some certain results for deferred Cesàro mean of double sequences. We recall some notations and basic definitions used in this paper.

By a convergence of double sequence we mean a convergence in Pringsheim's sense [18]. A double sequence $x = (x_{nm})$ is said to be convergent in the Pringsheim's sense if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that $|x_{nm} - L| < \varepsilon$ whenever $n, m \geq n_0$ [18]. In this case, we write $P - \lim_{n,m\to\infty} x_{nm} = L$.

A double sequence $x = (x_{nm})$ is bounded if there exist a positive number M such that $|x_{nm}| < M$ holds for all $(n, m) \in \mathbb{N} \times \mathbb{N} = \mathbb{N}^2$, i.e.,

$$\| x \|_{(\infty,2)} := \sup_{n,m} |x_{nm}| < \infty.$$

We will denote the set of all bounded double sequences by $\mathcal{M}_u$. Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded.

In [13], let $K \subset \mathbb{N}^2$ be a two-dimensional set of positive integers and let

$$K(n, m) := \{(j, k) \in K \ : \ (j, k) \leqslant (n, m)\}$$

where $(j, k) \leqslant (n, m)$ means that $j \leqslant n$ and $k \leqslant m$. Then, the lower asymptoic density of the set $K \subset \mathbb{N}^2$ is defined as

$$\underline{\delta}_2(K) := \liminf_{n,m\to\infty} \frac{|K(n, m)|}{mn}.$$

The vertical bars above indicate the cardinality of the set $K(n, m)$. In case the sequence $\left(\frac{|K(n,m)|}{mn}\right)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\delta_2(K) := \lim_{n,m\to\infty} \frac{|K(n, m)|}{mn}.$$

Following Mursaleen [13] we say that a double sequence $x = (x_{nm})$ is statistically convergent to the number L if for each $\epsilon > 0$,

$$\lim_{n,m\to\infty} \frac{1}{nm} |\{(j, k) : j \leq n, \ k \leq m, | \, x_{jk} - L \, | \geq \epsilon\}| = 0,$$

holds. In this case, we write $st_2 -\lim_{n,m\to\infty} x_{nm} = L$ and we denote the set of all double statistically convergent sequences by st_2.

Let $A = (a^{nm}_{jk})$ be a four dimensional summability matrix and $x = (x_{nm})$ be a double sequence. If $[Ax] := \{(Ax)_{nm}\}$ is convergent to L in Pringsheim sense then we say (x_{nm}) is $A-$summable to L where

$$(Ax)_{nm} := \sum_{j,k} a^{nm}_{jk} x_{jk} \ for \ all \ n, m \in \mathbb{N}$$

A said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit [9].

Recall that four dimensional Cesàro matrix $C_1 = (c_{jk}^{nm})$ is defined by

$$c_{jk}^{nm} = \begin{cases} \frac{1}{nm}, & j \leq n \text{ and } k \leq m \\ 0, & \text{otherwise.} \end{cases}$$

Let the index sequences $\lambda(n)$ and $\mu(m)$ be strictly increasing single valued sequences of positive integers and $x = (x_{nm})$ be a double sequence. Then, the Cesàro submethod $C_{\lambda,\mu} := (C_{\lambda,\mu}, 1, 1)$ is defined to be

$$(C_{\lambda,\mu}x)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{j=1,k=1}^{\lambda(n),\mu(m)} x_{jk}$$

where $\sum_{j=1,k=1}^{\lambda(n),\mu(m)} x_{jk} = \sum_{j=1}^{\lambda(n)} \sum_{k=1}^{\mu(m)} x_{jk}$. Since $\{(C_{\lambda,\mu}x)_{nm}\}$ is a subsequence of $\{(C_1x)_{nm}\}$, then the method $C_{\lambda,\mu}$ is RH-regular for any λ and μ [22].

2. MAIN RESULTS

In this section, the concepts of deferred Cesàro mean $D_{\beta,\gamma}$ and deferred statistical convergence for a double sequence $x = (x_{nm})$ are defined, and several theorems on this subject are given.

Throughout this paper $\beta(n) = q(n) - p(n)$, $\gamma(m) = r(m) - t(m)$ are represented by β and γ, respectively.

Definition 1. Let $x = (x_{kl})$ be a double sequence and $\beta(n) = q(n) - p(n)$, $\gamma(m) = r(m) - t(m)$. Then deferred Cesàro mean $D_{\beta,\gamma}$ of the double sequence x is defined by

$$\begin{aligned} (D_{\beta,\gamma}x)_{nm} &= \frac{1}{\beta(n)\ \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} x_{kl} \\ &= \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} x_{kl}, \end{aligned}$$

where $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$ are sequences of nonnegative integers satisfying the conditions $p(n) < q(n)$, $t(m) < r(m)$ and $\lim_{n\to\infty} q(n) = \infty$, $\lim_{m\to\infty} r(m) = \infty$.

We note that the method $D_{\beta,\gamma}$ is clearly regular for any choice of $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$.

Definition 2. Let $x = (x_{kl})$ be a double sequence and L be a real number. Then, the double sequence x is said to be $D_{\beta,\gamma}$-summable to L if

$$\lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} (x_{kl} - L) = 0$$

holds and it is denoted by $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} = L$ or $(\lim_{n,m\to\infty}(D_{\beta,\gamma}x)_{nm} = L)$.

Definition 3. Let K be a subset of $\mathbb{N}^2$ and denote the set

$$\{(k,l) : p(n) < k \leq q(n),\ t(m) < l \leq r(m),\ (k,\ l) \in K\}$$

by $K_{\beta,\gamma}(n,m)$. The deferred double natural density of K is defined by

$$\delta^{(2)}_{D_{\beta,\gamma}}(K) := \lim_{n,m\to\infty} \frac{1}{\beta(n)\gamma(m)} |K_{\beta,\gamma}(n,\ m)|$$

whenever the limit exists.

Also, because of $\delta^{(2)}_{D_{\beta,\gamma}}(K)$ does not exist for all $K \subset \mathbb{N}^2$, it is convenient to use upper deferred asymptotic density of K, defining by

$$\delta^{*(2)}_{D_{\beta,\gamma}}(K) := \limsup_{n,m\to\infty} \frac{|K_{\beta,\gamma}(n,m)|}{\beta(n)\ \gamma(m)}.$$

It is clear that, for the function $\delta^{*(2)}_{D_{\beta,\gamma}}$ the following axioms are hold for any $K \subset \mathbb{N}^2$:

i) if $\delta^{(2)}_{D_{\beta,\gamma}}(K)$ exists, then $\delta^{(2)}_{D_{\beta,\gamma}} = \delta^{*(2)}_{D_{\beta,\gamma}}(K)$,

ii) $\delta^{(2)}_{D_{\beta,\gamma}}(K) \neq 0$ if and only if $\delta^{*(2)}_{D_{\beta,\gamma}}(K) > 0$ and

iii) The function $\delta^{*(2)}_{D_{\beta,\gamma}}(K)$ is monotone increasing.

Definition 4. A double sequence $x = (x_{kl})$ is said to be deferred statistically convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1 \le k \le q(n),\ t(m)+1 \le l \le r(m),\ |x_{kl} - L| \ge \varepsilon\}|}{\beta(n)\ \gamma(m)} = 0$$

and it is denoted by $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$.

Using the above definitions, the next result follows immediately.

Theorem 2.1. *With* $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ *and* $\{t(m)\}$ *as in Definition 1, if* $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} = L$ *then,*

$$(D_{\beta,\gamma})\, st_2 - \lim_{n,m\to\infty} x_{nm} = L.$$

Proof. We assume that $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} = L$. Then for an arbitrary $\varepsilon > 0$ we have

$$\begin{aligned}
&\frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),\ r(m)} |x_{kl} - L| = \\
= \quad & \frac{1}{\beta(n)\ \gamma(m)} \left(\sum_{\substack{k=p(n)+1\\ l=t(m)+1\\ |x_{kl}-L|\ge\varepsilon}}^{q(n),\ r(m)} + \sum_{\substack{k=p(n)+1\\ l=t(m)+1\\ |x_{kl}-L|<\varepsilon}}^{q(n),\ r(m)} \right) |x_{kl} - L| \ge \\
\ge \quad & \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1\\ |x_{kl}-L|\ge\varepsilon}}^{q(n),\ r(m)} |x_{kl} - L| \ge \\
\ge \quad & \varepsilon . \frac{|\{(k,l) : p(n)+1 \le k \le q(n),\ t(m)+1 \le l \le r(m),\ |x_{kl} - L| \ge \varepsilon\}|}{\beta(n)\ \gamma(m)}.
\end{aligned}$$

Then, by taking limits when $n, m \to \infty$ we obtained

$$\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1 \le k \le q(n),\ t(m)+1 \le l \le r(m),\ |x_{kl} - L| \ge \varepsilon\}|}{\beta(n)\ \gamma(m)} = 0.$$

The proof is completed. □

Our next result is obtained from Theorem 2.1 easily.

Corollary 2.2. *If* $x_{nm} \to L(n, m \to \infty)$, *then* $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$.

Remark. *Generally, the converse of Theorem 2.1 and Corollary 2.2 are not true.*

For example; let $q(n)$ and $r(m)$ are strictly increasing sequences of positive integers and $h_1 \neq 0,\ h_2 \neq 0$ are arbitrary but fixed natural numbers. Let us define a double sequence $x = (x_{kl})$ for $n, m = 1, 2, \ldots$ as

$$x_{kl} := \begin{cases} k^2 l^2, & [\![\sqrt{q(n)}]\!] - h_1 < k \leq [\![\sqrt{q(n)}]\!],\ [\![\sqrt{r(m)}]\!] - h_2 < l \leq [\![\sqrt{r(m)}]\!] \\ 0, & \text{otherwise.} \end{cases}$$

If we consider $(D_{\beta,\gamma})$-method for a sequence $p(n)$ and $t(m)$ such that the condition

$$0 < p(n) \leq [\![\sqrt{q(n)}]\!] - h_1 \text{ and } 0 < t(m) \leq [\![\sqrt{r(m)}]\!] - h_2$$

holds. So, for an arbitrary $\varepsilon > 0$, we have

$$\begin{aligned} & \lim_{n,m\to\infty} \frac{|\{(k,l) : p(n) < k \leq q(n),\ t(m) < l \leq r(m),\ |x_{kl} - L| \geq \varepsilon\}|}{\beta(n)\ \gamma(m)} \\ = & \lim_{n,m\to\infty} \frac{h_1\ h_2}{\beta(n)\ \gamma(m)} = 0. \end{aligned}$$

It means that, $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$. On the other hand, we have

$$\begin{aligned} & \lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} |x_{kl} - L| \\ \geq & \lim_{n,m\to\infty} \frac{h_1 h_2 \Big(([\![\sqrt{q(n)}]\!] - h_1)\ ([\![\sqrt{r(m)}]\!] - h_2) \Big)^2}{\beta(n)\ \gamma(m)} = h_1 h_2. \end{aligned}$$

Given that, since $h_1 \neq 0$ and $h_2 \neq 0$, then $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} \neq L$ is obtained.

Theorem 2.3. *Let a bounded double sequence* $x = (x_{nm})$. *If* $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$, *then* $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} = L$.

Proof. We assume that $x = (x_{nm}) \in \mathcal{M}_u$ and $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$. In this case, there exists a real number $M > 0$ such that for all $n, m \in \mathbb{N}$ we have

$$|x_{nm} - L| \leq M.$$

For an arbitrary $\varepsilon > 0$, we have

$$\frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} |x_{kl} - L| =$$

$$= \frac{1}{\beta(n)\ \gamma(m)} \left(\sum_{\substack{k=p(n)+1 \\ l=t(m)+1 \\ |x_{kl}-L|\geq\varepsilon}}^{q(n),\ r(m)} + \sum_{\substack{k=p(n)+1 \\ l=t(m)+1 \\ |x_{kl}-L|<\varepsilon}}^{q(n),\ r(m)} \right) |x_{kl} - L| =$$

$$\leq \frac{1}{\beta(n)\ \gamma(m)} \left(M \sum_{\substack{k=p(n)+1 \\ l=t(m)+1 \\ |x_{kl}-L|\geq\varepsilon}}^{q(n),\ r(m)} 1 + \varepsilon \sum_{\substack{k=p(n)+1 \\ l=t(m)+1 \\ |x_{kl}-L|<\varepsilon}}^{q(n),\ r(m)} 1 \right) \leq$$

$$\leq M. \frac{|\{(k,l) : p(n) < k \leq q(n),\ t(m) < l \leq r(m),\ |x_{kl} - L| \geq \varepsilon\}|}{\beta(n)\ \gamma(m)} +$$

$$+ \varepsilon. \frac{|\{(k,l) : p(n) < k \leq q(n),\ t(m) < l \leq r(m),\ |x_{kl} - L| < \varepsilon\}|}{\beta(n)\ \gamma(m)}.$$

If

$$\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n) < k \leq q(n), t(m) < l \leq r(m),\ |x_{kl} - L| \geq \varepsilon\}|}{\beta(n)\ \gamma(m)} = 0$$

takes the limit when $n, m \to \infty$, then we have

$$\lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} |x_{kl} - L| = 0$$

and the proof is completed. □

Now, let us give following known Lemma that we will use it to prove of the Theorem 2.5

Lemma 2.4. *Let (a_{nm}) be a sequence of positive integers and $(k_n), (l_m)$ be increasing sequences of positive integers. If $\lim_{n,m\to\infty} a_{nm} = a \in \mathbb{R}$, then $\lim_{n,m\to\infty} a_{k_n l_m} = a$.*

Proof. Let $(a_{k_n l_m})$ be a subsequence of (a_{nm}) and let $\varepsilon > 0$ be given. Since $\lim_{n,m\to\infty} a_{nm} = a$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|a_{nm} - a| < \varepsilon$ for all $n, m \geq N$. Since $(k_n), (l_m)$ are increasing sequences of positive integers, we have $k_n \geq n,\ l_m \geq m,\ \forall n, m \in \mathbb{N}$. Hence, if $n, m \geq N$, then $k_n,\ l_m \geq N$ and $|a_{k_n l_m} - a| < \varepsilon$. That is, $\lim_{n,m\to\infty} a_{k_n l_m} = a$. □

Theorem 2.5. *Let $x = (x_{nm})$ be a real valued double sequence and $L \in \mathbb{R}$. If $st_2 - \lim_{n,m\to\infty} x_{nm} = L$, then $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$.*

Proof. Since $x = (x_{nm})$ is statistical convergent to L, then for $\forall\varepsilon > 0$,

$$\lim_{n,m\to\infty} \frac{1}{n\ m} |\{(k,\ l) \leq (n,\ m) : |x_{kl} - L| \geq \varepsilon\}| = 0.$$

By taking $a_{nm} = \frac{1}{nm}|\{(k,\ l) \le (n,\ m) : |x_{kl} - L| \ge \varepsilon\}|$ and $k_n = q(n),\ l_m = r(m)$ in the Lemma 2.4, we see that the following sequence converges to zero

$$\left\{\frac{|k \le q(n),\ l \le r(m) : |x_{kl} - L| \ge \varepsilon|}{q(n)\ r(m)}\right\}$$

i.e.,

$$\lim_{n,m\to\infty} \frac{1}{q(n)\ r(m)}|\{k \le q(n),\ l \le r(m) : |x_{kl} - L| \ge \varepsilon\}| = 0.$$

In this case, the inclusion

$$\begin{aligned} \{p(n) &< k \le q(n),\ t(m) < l \le r(m) : |x_{kl} - L| \ge \varepsilon\} \\ &\subseteq \{k \le q(n),\ l \le r(m) : |x_{kl} - L| \ge \varepsilon\} \end{aligned}$$

and the inequality

$$\begin{aligned} |\{p(n) &< k \le q(n),\ t(m) < l \le r(m) : |x_{kl} - L| \ge \varepsilon\}| \\ &\le |\{k \le q(n),\ l \le r(m) : |x_{kl} - L| \ge \varepsilon\}| \end{aligned}$$

hold for sequences $p(n) \le q(n),\ t(m) \le r(m),\ \forall n, m \in \mathbb{N}$. Hence, we have

$$\begin{aligned} &\lim_{n,m\to\infty} \frac{|\{p(n) < k \le q(n),\ t(m) < l \le r(m) : |x_{kl} - L| \ge \varepsilon\}|}{\beta(n)\ \gamma(m)} \\ \le &\lim_{n,m\to\infty} \frac{q(n)\ r(m)}{\beta(n)\ \gamma(m)} \frac{|\{k \le q(n),\ l \le r(m) : |x_{kl} - L| \ge \varepsilon\}|}{q(n)\ r(m)} \\ \le &\lim_{n,m\to\infty} \frac{q(n)\ r(m)}{q(n)\ r(m)(1 - \frac{t(m)}{r(m)} - \frac{p(n)}{q(n)} + \frac{p(n)\ t(m)}{q(n)\ r(m)})} \frac{|\{k \le q(n),\ l \le r(m) : |x_{kl} - L| \ge \varepsilon\}|}{q(n)\ r(m)} \\ \le &\lim_{n,m\to\infty} \frac{1}{(1 - \frac{t(m)}{r(m)} - \frac{p(n)}{q(n)} + \frac{p(n)\ t(m)}{q(n)\ r(m)})} \frac{|\{k \le q(n),\ l \le r(m) : |x_{kl} - L| \ge \varepsilon\}|}{q(n)\ r(m)}. \end{aligned}$$

Since

$$\liminf_{n,m\to\infty} \frac{1}{(1 - \frac{t(m)}{r(m)} - \frac{p(n)}{q(n)} + \frac{p(n)\ t(m)}{q(n)\ r(m)})} > 0$$

and

$$\lim_{n,m\to\infty} \frac{|\{k \le q(n),\ l \le r(m) : |x_{kl} - L| \ge \varepsilon\}|}{q(n)\ r(m)} = 0,$$

we can see $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$. □

Remark. *The converse of Theorem 2.5 is not true.*

For instance; if we define a sequence as

$$x_{nm} := \begin{cases} \frac{(n+1)\ (m+1)}{2}, & n \text{ and } m \text{ odd}, \\ -\frac{n\ m}{2}, & n \text{ or } m \text{ even} \end{cases}$$

and consider $p(n) = 2n,\ q(n) = 4n,\ t(m) = 2m,\ r(m) = 4m$. Then,

$$(D_{2n,\ 2m}) - \lim_{n,m\to\infty} x_{nm} = 0.$$

By Theorem 2.5 we get

$$(D_{2n,\ 2m})\ st_2 - \lim_{n,m\to\infty} x_{nm} = 0.$$

But, for all $\varepsilon > 0$,

$$\lim_{n,m\to\infty} \frac{|\{k \le n,\ l \le m : |x_{kl} - 0| \ge \varepsilon\}|}{n\ m} \ne 0$$

i.e., $st_2 - \lim_{n,m\to\infty} x_{nm} \ne 0$.

Corollary 2.6. *Under the conditions of Theorem 2.5, let $q(n),\ r(m)$ are sequences of positive integers such that $q(n) < n,\ r(m) < m$, for all $n, m \in \mathbb{N}$. Then $st_2 - \lim_{n,m\to\infty} x_{nm} = L$, implies $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$.*

Proof. Since $q(n) < n,\ r(m) < m$, for all $n, m \in \mathbb{N}$, the following inclusion

$$\begin{aligned} \{p(n) &< k \le q(n),\ t(m) < l \le r(m) : |x_{kl} - L| \ge \varepsilon\} \\ &\subseteq \{k \le n,\ l \le m : |x_{kl} - L| \ge \varepsilon\} \end{aligned}$$

and the inequality

$$\begin{aligned} |\{p(n) &< k \le q(n),\ t(m) < l \le r(m) : |x_{kl} - L| \ge \varepsilon\}| \\ &\le |\{k \le n,\ l \le m : |x_{kl} - L| \ge \varepsilon\}| \end{aligned}$$

hold. If we take limit when $n, m \to \infty$, we have

$$\begin{aligned} &\lim_{n,m\to\infty} \frac{|\{p(n) < k \le q(n),\ t(m) < l \le r(m) : |x_{kl} - L| \ge \varepsilon\}|}{\beta(n)\ \gamma(m)} \\ \le\ &\lim_{n,m\to\infty} \frac{n\ m}{\beta(n)\ \gamma(m)} \frac{|\{k \le n,\ l \le m : |x_{kl} - L| \ge \varepsilon\}|}{n\ m} \\ \le\ &\lim_{n,m\to\infty} \frac{n\ m}{q(n)\ r(m)\ (1 - \frac{t(m)}{r(m)} - \frac{p(n)}{q(n)} + \frac{p(n)}{q(n)} \frac{t(m)}{r(m)})} \frac{|\{k \le n,\ l \le m : |x_{kl} - L| \ge \varepsilon\}|}{n\ m} \\ \le\ &\lim_{n,m\to\infty} \left(\frac{1}{(1 - \frac{t(m)}{r(m)} - \frac{p(n)}{q(n)} + \frac{p(n)}{q(n)} \frac{t(m)}{r(m)})} \right) \left(\frac{|\{k \le n,\ l \le m : |x_{kl} - L| \ge \varepsilon\}|}{n\ m} \right). \end{aligned}$$

Since

$$\liminf_{n,m\to\infty} \frac{1}{(1 - \frac{t(m)}{r(m)} - \frac{p(n)}{q(n)} + \frac{p(n)}{q(n)} \frac{t(m)}{r(m)})} > 0$$

and then we have

$$\lim_{n,m\to\infty} \frac{|\{k \le n,\ l \le m : |x_{kl} - L| \ge \varepsilon\}|}{n\ m} = 0.$$

This gives that $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$. □

Theorem 2.7. *Let $x = (x_{nm})$ be a double sequence, $q(n) = n,\ r(m) = m$ for all $n, m \in \mathbb{N}$ and let $\{p(n)\},\ \{t(m)\}$ be arbitrary sequences. Then, $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$ if and only if $st_2 - \lim_{n,m\to\infty} x_{nm} = L$.*

Proof. Necessary: Let $q(n) = n,\ r(m) = m$ for all $n, m \in \mathbb{N}$ and $\{p(n)\},\ \{t(m)\}$ be arbitrary two sequences are given. We assume that $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$. We shall apply the same technique given in [1]. We define the sequences as

$$\begin{aligned} p_n &= n^{(1)} > p_{n^{(1)}} = n^{(2)} > p_{n^{(2)}} = n^{(3)} > \dots \\ t_m &= m^{(1)} > t_{m^{(1)}} = m^{(2)} > t_{m^{(2)}} = m^{(3)} > \dots \end{aligned}$$

for all $n, m \in \mathbb{N}$. The set $\{1 < k \leq n,\ 1 < l \leq m :\ |x_{kl} - L| \geq \varepsilon\}$ can be written as follows

$$\begin{aligned} & \{k \leq n,\ l \leq m :\ |x_{kl} - L| \geq \varepsilon\} \\ = & \left\{k \leq n^{(1)},\ l \leq m^{(1)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(1)} < k \leq n,\ m^{(1)} < l \leq m :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(1)} < k \leq n,\ l \leq m^{(1)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{k \leq n^{(1)},\ m^{(1)} < l \leq m :\ |x_{kl} - L| \geq \varepsilon\right\}. \end{aligned}$$

Hence, the sets can be written as follows respectively:

$$\begin{aligned} & \left\{k \leq n^{(1)},\ l \leq m^{(1)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ = & \left\{k \leq n^{(2)},\ l \leq m^{(2)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(2)} < k \leq n^{(1)},\ m^{(2)} < l \leq m^{(1)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(2)} < k \leq n^{(1)},\ l \leq m^{(2)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{k \leq n^{(2)},\ m^{(2)} < l \leq m^{(1)} :\ |x_{kl} - L| \geq \varepsilon\right\}, \end{aligned}$$

$$\begin{aligned} & \left\{n^{(1)} < k \leq n,\ l \leq m^{(1)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ = & \left\{n^{(1)} < k \leq n,\ l \leq m^{(2)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(1)} < k \leq n, m^{(2)} < \ l \leq m^{(1)} :\ |x_{kl} - L| \geq \varepsilon\right\} \end{aligned}$$

and

$$\begin{aligned} & \left\{k \leq n^{(1)},\ m^{(1)} < l \leq m :\ |x_{kl} - L| \geq \varepsilon\right\} \\ = & \left\{k \leq n^{(2)},\ m^{(1)} < l \leq m :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(1)} < k \leq n^{(2)},\ m^{(1)} < l \leq m :\ |x_{kl} - L| \geq \varepsilon\right\}. \end{aligned}$$

And the set $\left\{1 < k \leq n^{(2)},\ 1 < l \leq m^{(2)} :\ |x_{kl} - L| \geq \varepsilon\right\}$ can be written as

$$\begin{aligned} & \left\{k \leq n^{(2)},\ l \leq m^{(2)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ = & \left\{k \leq n^{(3)},\ l \leq m^{(3)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(3)} < k \leq n^{(2)}, m^{(3)} < \ l \leq m^{(2)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(3)} < k \leq n^{(2)},\ l \leq m^{(3)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{k \leq n^{(3)},\ m^{(3)} < \ l \leq m^{(2)} :\ |x_{kl} - L| \geq \varepsilon\right\}. \end{aligned}$$

Generally, the set $\left\{1 < k \leq n^{(h_1-1)},\ 1 < l \leq m^{(h_2-1)} :\ |x_{kl} - L| \geq \varepsilon\right\}$ is written

$$\begin{aligned} & \left\{1 < k \leq n^{(h_1-1)},\ 1 < l \leq m^{(h_2-1)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ = & \left\{k \leq n^{(h_1)},\ l \leq m^{(h_2)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(h_1)} < k \leq n^{(h_1-1)},\ m^{(h_2)} < l \leq m^{(h_2-1)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{n^{(h_1)} < k \leq n^{(h_1-1)},\ l \leq m^{(h_2)} :\ |x_{kl} - L| \geq \varepsilon\right\} \\ & \cup \left\{k \leq n^{(h_1)},\ \ m^{(h_2)} < l \leq m^{(h_2-1)} :\ |x_{kl} - L| \geq \varepsilon\right\} \end{aligned}$$

where $n^{(h_1)} \geq 1,\ \ n^{(h_1+1)} = 0$ and $m^{(h_2)} \geq 1,\ m^{(h_2+1)} = 0$ are hold for fixed $h_1, h_2 > 0$ positive integers. Therefore,

$$\frac{1}{nm} \left|\{k \leq n,\ l \leq m :\ |x_{kl} - L| \geq \varepsilon\}\right|$$

$$= \sum_{(i,j)=(0,0)}^{(h_1+1,h_2+1)} \frac{\Delta n^{(i)}\ \Delta m^{(j)}}{n\ m} \frac{\left|\left\{n^{(i+1)} < k \leq n^{(i)},\ m^{(j+1)} < l \leq m^{(j)} :\ |x_{kl} - L| \geq \varepsilon\right\}\right|}{\Delta n^{(i)}\ \Delta m^{(j)}}$$

is obtained, where $\Delta n^{(i)} := n^{(i)} - n^{(i+1)}$ and $\Delta m^{(j)} := m^{(j)} - m^{(j+1)}$. Moreover, since $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$, the sequence

$$\left\{ \frac{\left|\left\{n^{(i+1)} < k \leq n^{(i)},\ m^{(j+1)} < l \leq m^{(j)} :\ |x_{kl} - L| \geq \varepsilon\right\}\right|}{\Delta n^{(i)}\ \Delta m^{(j)}} \right\} \tag{2.1}$$

is convergent to zero for all $i, j \in \mathbb{N}$. If the matrix (b_{nmkl}) is defined as

$$b_{nmkl} := \begin{cases} \frac{\Delta n^{(i)}\ \Delta m^{(j)}}{n\ m}, & n^{(i+1)} < k \leq n^{(i)},\ m^{(j+1)} < l \leq m^{(j)},\ i, j = 1, 2, \ldots \\ 0, & \textit{otherwise}. \end{cases}$$

then the statistical convergence of the sequence $x = (x_{kl})$ is equivalent to the convergence of transform under the matrix (b_{nmkl}) of the sequence (2.1) . Since the matrix (b_{nmkl}) is regular,

$$\lim_{n,m\to\infty} \frac{1}{nm} \left|\{k \leq n,\ l \leq m :\ |x_{kl} - L| \geq \varepsilon\}\right| = 0$$

is obtained. So, the proof of theorem is completed. □

Corollary 2.8. *Let $x = (x_{nm})$ be a double sequence, $\{q(n)\}$ and $\{r(m)\}$ is equal to almost all positive integers. Then, $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$ for arbitrary sequences $\{p(n)\},\ \{t(m)\}$ implies $st_2 - \lim_{n,m\to\infty} x_{nm} = L$.*

Theorem 2.9. *Let $x = (x_{nm})$ be a double sequence, $\{q(n)\}$ and $\{r(m)\}$ be sequences of positive integers with $p(n) = n - 1$, $t(m) = m - 1$. In order that $(D_{\beta,\gamma})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$ implies $st_2 - \lim_{n,m\to\infty} x_{nm} = L$, it is necessary and sufficient that the sequences $\{q(n) - n\}$ and $\{r(m) - m\}$ be bounded.*

Proof. Necessary: Let $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$ implies $st_2 - \lim_{n,m\to\infty} x_{nm} = L$ and suppose that $\{q(n) - n\}$ and $\{r(m) - m\}$ are not bounded.

Let $n_1 = 1$; let n_2 be the smallest integer n for which $q(n) - n > q(n_1) - n_1$; let n_3 be the smallest integer n for which $q(n) - n > q(n_2) - n_2$. In general let $n_{\alpha+1}$ be the smallest integer n for which $q(n) - n > q(n_\alpha) - n_\alpha$. Then,

$$1 = n_1 < n_2 < n_3 < \ldots.$$

Similarly, Let $m_1 = 1$; let m_2 be the smallest integer m for which $r(m) - m > r(m_1) - m_1$; let m_3 be the smallest integer m for which $r(m) - m > r(m_2) - m_2$. In general let $m_{\beta+1}$ be the smallest integer m for which $r(m) - m > r(m_\beta) - m_\beta$. Then,

$$1 = m_1 < m_2 < m_3 < \ldots.$$

We find that we can set

$$x\left(q_{n_i}\ r_{m_j}\right) = q_{n_i}\ r_{m_j},\ i, j = 1, 2, \ldots$$

and choose the remaining elements of a sequence

$$\{x(nm)\} \equiv \{x_{nm}\}$$

so that the sequence $\{x_{nm}\}$ is summable $(D_{\beta,\gamma})$ to L, $i.e$,$(D_{\beta,\gamma})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$. But $\{x_{nm}\}$ cannot be summable $(C, 1, 1)$ since the condition

$$\lim_{n,m\to\infty} \frac{1}{nm} \left|\{k \le n,\ l \le m :\ |x_{kl} - L| \ge \varepsilon\}\right| \ne 0.$$

This contradicts the hypothesis. Therefore, the sequences $\{q(n) - n\}$ and $\{r(m) - m\}$ are bounded.

Sufficient: Since the sequences $\{q(n) - n\}$ and $\{r(m) - m\}$ are bounded, there exists $\exists L_1, L_2 \in \mathbb{R}$ such that $q(n) - n = L_1$ and $r(m) - m = L_2$ for all $n, m \in \mathbb{N}$. Hence, $\{q(n)\}$ and $\{r(m)\}$ is equal to almost all positive integers. If Corollary 2.8 is considered, the proof is completed. □

Theorem 2.10. *Let $x = (x_{nm})$ be a double sequence. In order that $(D_{\beta,\gamma})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$ implies $st_2 - \lim_{n,m\to\infty} x_{nm} = L$ where $p(n) = n - 1$, $t(m) = m - 1$ and*

$$q(n) := \begin{cases} h_{i+1} - 1, & n = h_i,\ i = 1, 2, \ldots \\ n, & \text{otherwise}, \end{cases}$$

$$r(m) := \begin{cases} s_{j+1} - 1, & m = s_j,\ j = 1, 2, \ldots \\ m, & \text{otherwise}. \end{cases}$$

$\{h_n\}, \{s_m\}$ being increasing sequences of integers for which $h_n > n$, $s_m > m$, it is necessary and sufficient that the sequences $\left\{\frac{q(n)}{n}\right\}$ and $\left\{\frac{r(m)}{m}\right\}$ be bounded.

Proof. Necessary: Let $(D_{\beta,\gamma})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$. Corresponding to each index (n, m), let $i = i(n)$, $j = j(m)$ be such that $h_i \le n < h_{i+1}$ and $s_j \le m <$

s_{j+1}. Then, the equality

$$
\begin{aligned}
& \frac{1}{nm}\left|\{k\leq n,\ l\leq m:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
= & \frac{1}{nm}\left|\{k\leq h_1-1,\ l\leq s_1-1:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
& +\frac{1}{nm}\left|\{h_1\leq k\leq h_2-1,\ l\leq s_1-1:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
& +\frac{1}{nm}\left|\{k\leq h_1-1,\ s_1\leq l\leq s_2-1:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
& +\frac{1}{nm}\left|\{h_1\leq k\leq h_2-1,\ s_1\leq l\leq s_2-1:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
& +\cdots+\frac{1}{nm}\left|\{h_i\leq k\leq h_{i+1}-1,\ s_j\leq l\leq s_{j+1}-1:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
& -\frac{1}{nm}\left|\{n+1\leq k\leq h_{i+1}-1,\ m+1\leq l\leq s_{j+1}-1:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
= & \frac{1}{nm}\left|\{k\leq 1,\ l\leq 1:\ |x_{kl}-L|\geq\varepsilon\}\right|+\frac{1}{nm}\left|\{1<k\leq 2,\ l\leq 1:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
& +\frac{1}{nm}\left|\{k\leq 1,\ 1<l\leq 2:\ |x_{kl}-L|\geq\varepsilon\}\right|+\frac{1}{nm}\left|\{1<k\leq 2,\ 1<l\leq 2:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
& +\cdots \\
& +\frac{(h_2-h_1)(s_2-\ s_1)}{nm} \\
& \times\frac{\left|\{h_1-1<k\leq h_2-1,\ s_1-1<l\leq s_2-1:\ |x_{kl}-L|\geq\varepsilon\}\right|}{(h_2-h_1)(s_2-\ s_1)} \\
& +\cdots \\
& +\frac{(h_{i+1}-h_i)(s_{j+1}-s_j)}{nm} \\
& \times\frac{\left|\{h_i-1<k\leq h_{i+1}-1,\ s_j-1<l\leq s_{j+1}-1:\ |x_{kl}-L|\geq\varepsilon\}\right|}{(h_{i+1}-h_i)(s_{j+1}-s_j)} \\
& -\frac{1}{nm}\left|\{n+1<k\leq n+2,\ m+1<l\leq m+2:\ |x_{kl}-L|\geq\varepsilon\}\right| \\
& -\cdots-\frac{1}{nm}\left|\{h_{i+1}-1<k\leq h_{i+1},\ s_{j+1}-1<l\leq s_{j+1}:\ |x_{kl}-L|\geq\varepsilon\}\right|
\end{aligned}
$$

is hold. Therefore, statistical convergence of the sequence $x=(x_{nm})$ is equavalent to the convergence of transform under the matrix

$$
b_{nmkl}:=\begin{cases} \frac{(h_{i+1}-h_i)\ (s_{j+1}-s_j)}{nm}, & \begin{array}{c} h_i-1<k\leq h_{i+1}-1, \\ s_j-1<l\leq s_{j+1}-1,\ i,j=1,2,\ldots \end{array} \\ nm, & \textit{otherwise} \end{cases}
$$

of the sequence

$$
\left\{\frac{\left|\{h_i-1<k\leq h_{i+1}-1,\ s_j-1<l\leq s_{j+1}-1:\ |x_{kl}-L|\geq\varepsilon\}\right|}{(h_{i+1}-h_i)(s_{j+1}-s_j)}\right\}.
$$

The transform satisfies the conditions of regularity when and only when the sequences

$$
\left\{\frac{2h_{i+1}-n-2}{n}\right\}\quad\text{and}\quad\left\{\frac{2s_{j+1}-m-2}{m}\right\}
$$

are bounded for all (n,m) i.e., when and only when the sequences $\left\{\frac{h_{i+1}}{n}\right\}$ and $\left\{\frac{s_{j+1}}{m}\right\}$ are bounded for all (n,m) and therefore, when and only when the sequences $\left\{\frac{q(n)}{n}\right\}$ and $\left\{\frac{r(m)}{m}\right\}$ are bounded for all (n,m). □

3. SOME INCLUSION RESULTS FOR CESÀRO SUBMETHODS

Let $\lambda = (\lambda(n))$ and $\mu = (\mu(m))$ are strictly increasing sequences of positive integers such that $\lambda(0) = 0$ and $\mu(0) = 0$. Then, $D_{\beta,\gamma}$-statistically convergence is defined as $D_{\lambda,\mu}$-statistically convergence taking

$$q(n) = \lambda(n),\ p(n) = \lambda(n-1),\ r(m) = \mu(m) \text{ and } t(m) = \mu(m-1).$$

It is denoted by $(D_{\lambda,\mu})\ st_2$.

A double sequence $x = (x_{jk})$ is said to be A-statistically convergent to L, written $(A)\, st_2 - \lim_{n,m\to\infty} x_{nm} = L$ [15], if $\delta_A^{(2)}(K_\varepsilon) = 0$ for every $\varepsilon > 0$, where

$$K_\varepsilon := \{\ k \le n,\ l \le m :\ |x_{kl} - L| \geqslant \varepsilon\}.$$

We examine the concept of $C_{\lambda,\mu}$-statistically convergence and its relations to $D_{\lambda,\mu}$-statistically convergence. Now, we define $C_{\lambda,\mu}$-statistically convergence, note that if $A = C_{\lambda,\mu}$, then $(C_{\lambda,\mu})\, st_2 - \lim_{n,m\to\infty} x_{nm} = L$ if for every $\varepsilon > 0$,

$$\begin{aligned} \delta^{(2)}_{C_{\lambda,\mu}}(K_\varepsilon) &= \lim_{n,m\to\infty} \frac{1}{\lambda(n)\mu(m)} \left(C_{\lambda,\mu} \cdot \chi_{K_\varepsilon}\right)_{nm} \\ &= \lim_{n,m\to\infty} \frac{1}{\lambda(n)\mu(m)} \sum_{k=1,l=1}^{\lambda(n),\mu(m)} \chi_{K_\varepsilon}(k,l) \\ &= \lim_{n,m\to\infty} \frac{1}{\lambda(n)\mu(m)} |\{k \le \lambda(n),\ l \le \mu(m) : |x_{kl} - L| \ge \varepsilon\}| = 0. \end{aligned}$$

The following theorem gives a relation between $C_{\lambda,\mu}$-statistically convergence and $D_{\lambda,\mu}$-statistically convergence.

Theorem 3.1. *Let $\lambda = (\lambda(n))$ and $\mu = (\mu(m))$ are sequences strictly increasing of positive integers such that $\lambda(0) = 0$ and $\mu(0) = 0$. If $(D_{\lambda,\mu})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$, then $(C_{\lambda,\mu})\, st_2 - \lim_{n,m\to\infty} x_{nm} = L$.*

Proof. We assume that $(D_{\lambda,\mu})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$. For arbitrary $\varepsilon > 0$,

$$
\begin{aligned}
& (C_{\lambda,\mu}x)_{nm} := \frac{1}{\lambda(n)\ \mu(m)} |\{k \leq \lambda(n),\ \ l \leq \mu(m) :\ |x_{kl} - L| \geq \varepsilon\}| \\
= \quad & \frac{(\lambda(1)-\lambda(0))\ (\mu(1)-\mu(0))}{\lambda(n)\ \mu(m)} \frac{|\{\lambda(0) < k \leq \lambda(1),\ \mu(0) < l \leq \mu(1) :\ |x_{kl} - L| \geq \varepsilon\}|}{(\lambda(1)-\lambda(0))\ (\mu(1)-\mu(0))} \\
+ \quad & \dots \\
+ \quad & \frac{\Delta\lambda(k)\ (\mu(1)-\mu(0))}{\lambda(n)\ \mu(m)} \frac{|\{\lambda(k-1) < k \leq \lambda(k),\ \mu(0) < l \leq \mu(1) :\ |x_{kl} - L| \geq \varepsilon\}|}{\Delta\lambda(k)\ (\mu(1)-\mu(0))} \\
+ \quad & \dots \\
+ \quad & \frac{(\lambda(1)-\lambda(0))\ \Delta\mu(l)}{\lambda(n)\ \mu(m)} \frac{|\{\lambda(0) < k \leq \lambda(1),\ \mu(l-1) < l \leq \mu(l) :\ |x_{kl} - L| \geq \varepsilon\}|}{(\lambda(1)-\lambda(0))\ \Delta\mu(l)} \\
+ \quad & \dots \\
+ \quad & \frac{\Delta\lambda(k)\ \Delta\mu(l)}{\lambda(n)\ \mu(m)} \frac{|\{\lambda(k-1) < k \leq \lambda(k),\ \mu(l-1) < l \leq \mu(l) :\ |x_{kl} - L| \geq \varepsilon\}|}{\Delta\lambda(k)\ \Delta\mu(l)} \\
+ \quad & \dots \\
+ \quad & \frac{\Delta\lambda(n)\ \Delta\mu(m)}{\lambda(n)\ \mu(m)} \frac{|\{\lambda(n-1) < k \leq \lambda(n), \mu(m-1) < l \leq \mu(m) :\ |x_{kl} - L| \geq \varepsilon\}|}{\Delta\lambda(n)\ \Delta\mu(m)}
\end{aligned}
$$

is obtained, where $\Delta\lambda(n) = \lambda(n) - \lambda(n-1)$ and $\Delta\mu(m) = \mu(m) - \mu(m-1)$. If we say

$$(DS_{\lambda,\mu}x)_{kl} := \frac{1}{\Delta\lambda(k)\ \Delta\mu(l)} |\{\lambda(k-1) < i \leq \lambda(k),\ \mu(l-1) < j \leq \mu(l) :\ |x_{ij} - L| \geq \varepsilon\}|,$$

we can obtain

$$
\begin{aligned}
& (C_{\lambda,\mu}x)_{nm} \\
= \quad & \frac{(\lambda(1)-\lambda(0))\ (\mu(1)-\mu(0))}{\lambda(n)\ \mu(m)}\ (DS_{\lambda,\mu}x)_{11} + \frac{(\lambda(2)-\lambda(1))\ (\mu(1)-\mu(0))}{\lambda(n)\ \mu(m)}\ (DS_{\lambda,\mu}x)_{21} \\
+ \quad & \frac{(\lambda(1)-\lambda(0))\ (\mu(2)-\mu(1))}{\lambda(n)\ \mu(m)}\ (DS_{\lambda,\mu}x)_{12} + \dots + \frac{\Delta\lambda(n)\ \Delta\mu(m)}{\lambda(n)\ \mu(m)}\ (DS_{\lambda,\mu}x)_{nm}.
\end{aligned}
$$

Let $B = (b_{nmkl})$ be the matrix defined by

$$b_{nmkl} := \begin{cases} \frac{\Delta\lambda(k)\ \Delta\mu(l)}{\lambda(n)\ \mu(m)}, & k \leq n,\ l \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, since B satisfies conditions of Theoren II in [19] B is regular and we see that $(C_{\lambda,\mu}x)_{nm} = (B(DS_{\lambda,\mu}))_{nm}$. Hence since $(D_{\lambda,\mu})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$, we get $(C_{\lambda,\mu})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$. This completes the proof of the theorem. □

Theorem 3.2. *The method $D_{\beta,\gamma}$ includes $(C,1,1)$ if and only if the sequence $\left\{\frac{p(n)\ t(m)}{\beta(n)\ \gamma(m)}\right\}$ is bounded.*

Proof. Let $x = (x_{kl})$ be a given double sequence; then for any transformation $(D_{\beta,\gamma})$, we have

$$
\begin{aligned}
(D_{\beta,\gamma}x)_{nm} &= \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),\ r(m)} x_{kl} \\
&= \frac{1}{\beta(n)\ \gamma(m)} \left(\sum_{(k,l)=(1,1)}^{q(n),\ r(m)} x_{kl} - \sum_{(k,l)=(1,1)}^{p(n),\ r(m)} x_{kl} - \sum_{(k,l)=(1,1)}^{q(n),\ t(m)} x_{kl} + \sum_{(k,l)=(1,1)}^{p(n),\ t(m)} x_{kl} \right) \\
&= \frac{q(n)\ r(m)}{\beta(n)\ \gamma(m)} \left(\frac{1}{q(n)\ r(m)} \sum_{(k,l)=(1,1)}^{q(n),\ r(m)} x_{kl} \right) - \frac{p(n)\ r(m)}{\beta(n)\ \gamma(m)} \left(\frac{1}{p(n)\ r(m)} \sum_{(k,l)=(1,1)}^{p(n),\ r(m)} x_{kl} \right) \\
&- \frac{q(n)\ t(m)}{\beta(n)\ \gamma(m)} \left(\frac{1}{q(n)\ t(m)} \sum_{(k,l)=(1,1)}^{q(n),\ t(m)} x_{kl} \right) + \frac{p(n)\ t(m)}{\beta(n)\ \gamma(m)} \left(\frac{1}{p(n)\ t(m)} \sum_{(k,l)=(1,1)}^{p(n),\ t(m)} x_{kl} \right).
\end{aligned}
$$

That is,

$$
\begin{aligned}
(D_{\beta,\gamma}x)_{nm} &= \frac{q(n)\ r(m)}{\beta(n)\ \gamma(m)} (C_{q,r}x)_{nm} - \frac{p(n)\ r(m)}{\beta(n)\ \gamma(m)} (C_{p,r}x)_{nm} \\
&\quad - \frac{q(n)\ t(m)}{\beta(n)\ \gamma(m)} (C_{q,t}x)_{nm} + \frac{p(n)\ t(m)}{\beta(n)\ \gamma(m)} (C_{p,t}x)_{nm}. \qquad (3.1)
\end{aligned}
$$

Therefore, the sequence (C_1x) may be partitioned into two disjoint subsequences, namely $(C_{\beta,\gamma}x)_{nm} = (C_1x)_{\beta(n)\gamma(m)}$. Let us define the matrix $B = (b_{nmkl})$ as

$$
b_{nmkl} = \begin{cases} \frac{q(n)\ r(m)}{\beta(n)\ \gamma(m)} &, \quad k = q(n), l = r(m) \\ \frac{p(n)\ r(m)}{\beta(n)\ \gamma(m)} &, \quad k = p(n), l = r(m) \\ \frac{q(n)\ t(m)}{\beta(n)\ \gamma(m)} &, \quad k = q(n), l = t(m) \\ \frac{p(n)\ t(m)}{\beta(n)\ \gamma(m)} &, \quad k = p(n), l = t(m) \\ 0 &, \quad \textit{otherwise.} \end{cases}
$$

Regarding (3.1) as a transformation of the form

$$
\sum_{(1,1)}^{(\infty,\infty)} b_{nmkl}\ x_{kl}
$$

which carries $C_1 = (C, 1, 1)$ into $D_{\beta,\gamma}$, we see that (3.1) satisfies the conditions (a), (b), (c), (d) and (e) of Theorem II in Robison [19] and that (3.1) satisfies (f) when and only when

$$
\begin{aligned}
&\frac{q(n)\ r(m)}{\beta(n)\ \gamma(m)} + \frac{p(n)\ r(m)}{\beta(n)\ \gamma(m)} + \frac{q(n)\ t(m)}{\beta(n)\ \gamma(m)} + \frac{p(n)\ t(m)}{\beta(n)\ \gamma(m)} \\
= \ &\frac{(q(n) + p(n))\ (r(m) + t(m))}{\beta(n)\ \gamma(m)}
\end{aligned}
$$

is bounded. □

The theorems below give us some inclusion results between $D_{\beta,\gamma}$ and $C_{\lambda,\mu}$ methods.

Theorem 3.3. *Let $\{\lambda(n)\}$ and $\{\mu(m)\}$ be infinite subset of $\mathbb{N}$ with $\lambda(0)=0$ and $\mu(0)=0$. Then $C_{\lambda,\mu}$ includes $D_{\beta,\gamma}$.*

Proof. We shall apply the same technique found in [1]. Assume $x=(x_{nm})$ is a double sequence that is $D_{\beta,\gamma}$-summable to L . Then, for any $n,m\in\mathbb{N}$

$$
\begin{aligned}
&(C_{\lambda,\mu}x)_{nm} = \frac{1}{\lambda(n)\ \mu(m)} \sum_{(k,l)=(1,1)}^{\lambda(n),\ \mu(m)} x_{kl}\\
= \quad & \frac{(\lambda(1)-\lambda(0))\ (\mu(1)-\mu(0))}{\lambda(n)\ \mu(m)} \left(\frac{1}{(\lambda(1)-\lambda(0))\ (\mu(1)-\mu(0))} \sum_{(k,l)=(1,1)}^{\lambda(1)\ ,\mu(1)} x_{kl} \right)\\
+ \quad & \frac{(\lambda(2)-\lambda(1))\ (\mu(1)-\mu(0))}{\lambda(n)\ \mu(m)} \left(\frac{1}{(\lambda(2)-\lambda(1))\ (\mu(1)-\mu(0))} \sum_{(k,l)=(\lambda(1)+1,1)}^{\lambda(2),\ \mu(1)} x_{kl} \right)\\
+ \quad & \dots\\
+ \quad & \frac{\Delta\lambda(n)\ (\mu(1)-\mu(0))}{\lambda(n)\ \mu(m)} \left(\frac{1}{\Delta\lambda(n)\ (\mu(1)-\mu(0))} \sum_{(k,l)=(\lambda(n-1)+1,1)}^{\lambda(n)\ ,\mu(1)} x_{kl} \right)\\
+ \quad & \frac{(\lambda(2)-\lambda(1))\ (\mu(2)-\mu(1))}{\lambda(n)\ \mu(m)} \left(\frac{1}{(\lambda(2)-\lambda(1))\ (\mu(2)-\mu(1))} \sum_{(k,l)=(\lambda(1)+1,\mu(1)+1)}^{\lambda(2),\ \mu(2)} x_{kl} \right)\\
+ \quad & \dots\\
+ \quad & \frac{\Delta\lambda(n)\ \Delta\mu(m)}{\lambda(n)\ \mu(m)} \left(\frac{1}{\Delta\lambda(n)\ \Delta\mu(m)} \sum_{\substack{k=\lambda(n-1)+1\\ l=\mu(m-1)+1}}^{\lambda(n),\ \mu(m)} x_{kl} \right)\\
= \quad & \frac{(\lambda(1)-\lambda(0))\ (\mu(1)-\mu(0))}{\lambda(n)\ \mu(m)} (D_{\beta,\gamma}x)_{11} + \dots + \frac{\Delta\lambda(n)\ \Delta\mu(m)}{\lambda(n)\ \mu(m)} (D_{\beta,\gamma}x)_{nm}
\end{aligned}
$$

Let $B=(b_{nmkl})$ be the matrix defined by

$$
b_{nmkl} := \begin{cases} \frac{\Delta\lambda(k)\ \Delta\mu(l)}{\lambda(n)\ \mu(m)}, & k=1,2,\dots,n\ ,\ l=1,2,\dots,m\\ 0, & \textit{otherwise}. \end{cases}
$$

Clearly, since B satisfies conditions of Theoren II in [19] B is regular, and we see that $(C_{\lambda,\mu}x)_{nm}=(B(D_{\beta,\gamma}x))_{nm}$. Since

$$
\lim_{n,m\to\infty}(D_{\beta,\gamma}x)_{nm} = L \text{ and } B \text{ is regular, then}
$$

we have $\lim_{n,m\to\infty}(B(D_{\beta,\gamma}x))_{nm}=L.$ Hence, $\lim_{n,m\to\infty}(C_{\lambda,\mu}x)_{nm}=L$, and $C_{\lambda,\mu}$ includes $D_{\beta,\gamma}$. □

Theorem 3.4. *Let $\{\lambda(n)\}$ and $\{\mu(m)\}$ be infinite subsets of $\mathbb{N}$ with $\lambda(0)=0$, $\mu(0)=0$. Then $D_{\beta,\gamma}$ includes $C_{\lambda,\mu}$ if and only if $\liminf_{n\to\infty}\frac{\lambda(n)}{\lambda(n-1)}>1$ and $\liminf_{m\to\infty}\frac{\mu(m)}{\mu(m-1)}>1$.*

Proof. Again, we shall apply the same technique found in [1]. Let $x = (x_{nm})$ be a given double sequence. Then

$$\begin{aligned}
(D_{\beta,\gamma}x)_{nm} &= \frac{1}{\Delta\lambda(n)\ \Delta\mu(m)} \sum_{(k,\ l)=\lambda(n-1)+1,\ \mu(m-1)+1}^{\lambda(n),\ \mu(m)} x_{kl} \\
&= \frac{\lambda(n)\ \mu(m)}{\Delta\lambda(n)\ \Delta\mu(m)} \left(\frac{1}{\lambda(n)\ \mu(m)} \sum_{(k,l)=(1,1)}^{\lambda(n),\ \mu(m)} x_{kl} \right) \\
&- \frac{\lambda(n)\ \mu(m-1)}{\Delta\lambda(n)\ \Delta\mu(m)} \left(\frac{1}{\lambda(n)\ \mu(m-1)} \sum_{(k,l)=(1,1)}^{\lambda(n),\ \mu(m-1)} x_{kl} \right) \\
&+ \frac{\lambda(n-1)\ \mu(m-1)}{\Delta\lambda(n)\ \Delta\mu(m)} \left(\frac{1}{\lambda(n-1)\ \mu(m-1)} \sum_{(k,l)=(1,1)}^{\lambda(n-1),\ \mu(m-1)} x_{kl} \right) \\
&- \frac{\lambda(n-1)\ \mu(m)}{\Delta\lambda(n)\ \Delta\mu(m)} \left(\frac{1}{\lambda(n-1)\ \mu(m)} \sum_{(k,l)=(1,1)}^{\lambda(n-1),\ \mu(m)} x_{kl} \right) \\
&= \frac{\lambda(n)\ \mu(m)}{\Delta\lambda(n)\ \Delta\mu(m)} (C_{\lambda,\mu}x)_{nm} - \frac{\lambda(n)\ \mu(m-1)}{\Delta\lambda(n)\ \Delta\mu(m)} (C_{\lambda,\mu}x)_{n,m-1} \\
&+ \frac{\lambda(n-1)\ \mu(m-1)}{\Delta\lambda(n)\ \Delta\mu(m)} (C_{\lambda,\mu}x)_{n-1,m-1} - \frac{\lambda(n-1)\ \mu(m)}{\Delta\lambda(n)\ \Delta\mu(m)} (C_{\lambda,\mu}x)_{n-1,m}.
\end{aligned}$$

Let $B = (b_{nmkl})$ be the matrix defined by

$$b_{nmkl} := \begin{cases} \frac{\lambda(n)\ \mu(m)}{\Delta\lambda(n)\ \Delta\mu(m)}, & k = \lambda(n),\ l = \mu(m) \\ -\frac{\lambda(n)\ \mu(m-1)}{\Delta\lambda(n)\ \Delta\mu(m)}, & k = \lambda(n),\ l = \mu(m-1) \\ \frac{\lambda(n-1)\ \mu(m-1)}{\Delta\lambda(n)\ \Delta\mu(m)}, & k = \lambda(n-1),\ l = \mu(m-1) \\ -\frac{\lambda(n-1)\ \mu(m)}{\Delta\lambda(n)\ \Delta\mu(m)}, & k = \lambda(n-1),\ l = \mu(m) \\ 0, & \textit{otherwise}. \end{cases}$$

Thus, $(D_{\beta,\gamma}x)_{nm} = (B\,(C_{\lambda,\mu}x))_{nm}$, and hence $D_{\beta,\gamma}$ will include $C_{\lambda,\mu}$ if and only if B is regular. Clearly, B satisfies conditions of Theoren II in [19]. Thus B will be regular if and only if the sequence

$$\left\{ \frac{\lambda(n)\ \mu(m)}{\Delta\lambda(n)\ \Delta\mu(m)} + \frac{\lambda(n)\ \mu(m-1)}{\Delta\lambda(n)\ \Delta\mu(m)} + \frac{\lambda(n-1)\ \mu(m-1)}{\Delta\lambda(n)\ \Delta\mu(m)} + \frac{\lambda(n-1)\ \mu(m)}{\Delta\lambda(n)\ \Delta\mu(m)} \right\}$$

is bounded. But,

$$\begin{aligned}
&\frac{(\lambda(n)+\lambda(n-1))\ \ (\mu(m)+\mu(m-1))}{\Delta\lambda(n)\ \Delta\mu(m)} \\
&= \frac{(\lambda(n)+\lambda(n-1))\ \ (\mu(m)+\mu(m-1))}{(\lambda(n)-\lambda(n-1))\ (\mu(m)-\mu(m-1))} \\
&= \left(1 + \frac{2\lambda(n-1)}{\lambda(n)-\lambda(n-1)}\right) \left(1 + \frac{2\mu(m-1)}{\mu(m)-\mu(m-1)}\right) \\
&= \left(1 + \frac{2}{\frac{\lambda(n)}{\lambda(n-1)} - 1}\right) \left(1 + \frac{2}{\frac{\mu(m)}{\mu(m-1)} - 1}\right)
\end{aligned}$$

and the last expression is bounded if and only if $\liminf_{n\to\infty} \frac{\lambda(n)}{\lambda(n-1)} > 1$ and $\liminf_{m\to\infty} \frac{\mu(m)}{\mu(m-1)} > 1$. The proof is completed. □

Combining Theorems 3.3 and 3.4, the following theorem becomes evident.

Theorem 3.5. *Let $\{\lambda(n)\}$ and $\{\mu(m)\}$ be infinite subsets of $\mathbb{N}$ with $\lambda(0) = 0$, $\mu(0) = 0$. Then $D_{\beta,\gamma}$ is equivalent to $C_{\lambda,\mu}$ if and only if $\liminf_{n\to\infty} \frac{\lambda(n)}{\lambda(n-1)} > 1$ and $\liminf_{m\to\infty} \frac{\mu(m)}{\mu(m-1)} > 1$.*

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İ. Dağadur, Department of Mathematics, Faculty of Sciences and Literature, Mersin University, Mersin, 33343 TURKEY
E-mail address: ilhandagadur@yahoo.com

Ş. Sezgek, Department of Mathematics, Faculty of Science and Literature, Mersin University, Mersin, 33343 TURKEY
E-mail address: seydasezgek@mersin.edu.tr