Fermionic Particle Production by Varying Electric and Magnetic Fields*

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Abstract *Creation of fermionic particles by a time-dependent electric field and a space-dependent magnetic field is studied with the Bogoulibov transformation method. Exact analytic solutions of the Dirac equation are obtained in terms of the Whittaker functions and the particle creation number density depending on the electric and magnetic fields is determined.*

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1 Introduction

There has been an increasing interest in the mechanism of pair creation by strong electric fields after the pioneer studies of Sauter,[1] Heisenberg and Euler[2] and Schwinger.[3] Since then, the creation of scalar and fermionic particles from the unstable vacuum in the presence of external electric fields become a very important phenomenon in the quantum electrodynamics (QED). The widely used formula in these studies for calculation of the particle-creation rate is derived by Schwinger using the proper-time method and given as (in natural units, $\hbar = c = 1$):[3]

$$\omega = \frac{(eE)^2}{4\pi^3} \sum_{n=1}^{+\infty} \frac{1}{n^2} \exp\Big(-\frac{n\pi m^2}{eE}\Big), \tag{1}$$

where E is electric field and m and e are the mass and charge of the electron. This formula ignores the collective interactions of the created particles and their effect to the initial field. Based on this formula, the particle-creation process in external electric field is appreciable above the critical value of electric field, $E_{\rm cr} \simeq 10^{16}$ V/cm. The critical electric fields can be attainable in the modern particle colliders in heavy-ion collisions[4] and in the X-ray electron laser facilities.[5] Moreover, according to the recent studies such strong electric fields can be generated at the surface of the quark and neutron stars.[6]

After the discovery of the Higgs particle we ensure that the vacuum is not a zero-field state and it can be fluctuated by external field sources. Since the instability of the vacuum is generated by the contraction or expansion of the spacetime, this mechanism has started to play an important role also in cosmology to understand for example the source of the considerable entropy of the present universe.[7–8]

Particle creation in the presence of strong electric fields have been discussed by different authors and recently it has become one of the most studied phenomena of the QED. Among the many authors,[9–17] Kleinert *et al.*[10] studied fermion creation in space- or time-dependent electric fields via semiclassical approximation. Kim and Page[11] have studied Schwinger pair production mechanism in spinor and scalar QED for space-dependent electric and magnetic fields by applying the instanton method. Kluger *et al.*[10] have studied the fermion pair production by using adiabatic regularization method. Also, Tanji[4] has studied the Klein–Gordon and Dirac equations with the use of a damping electric field for investigating numerically the particle creation related to instantaneous positive and negative frequency solutions and discussed the time evaluation of the distributions.

In this paper we study the fermionic particle creation by considering combined electric and magnetic fields, that is a different configuration from the above mentioned studies. The Dirac equation is solved exactly in the existence of combined exponentially damping electric field and space-dependent magnetic field. These fields are constructed from a 4-vector electromagnetic potential

$$A_\varepsilon = \Big[-\frac{B_0}{\nu}\,{\rm e}^{-\nu x}\delta_\varepsilon^2 + \frac{E_0}{\xi}\,{\rm e}^{-\xi t}\delta_\varepsilon^3\Big], \tag{2}$$

where B_0, E_0, ν and ξ are constants and the Greek index $\varepsilon = 0,1,2,3$ belonging to time t and cartesian space coordinates x, y, z, respectively.

The form of the electric field represents a definite situation as it tends to zero for $t \to \infty$. In the literature, some authors conclude that there is an essential relation between the electrodynamics of a constant field and thermodynamics.[9] Spokoiny has shown that if the electric field derived from Eq. (2) is the asymptotic of the damping

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field at $t \to \infty$, particles can be produced with a thermal spectrum. In the present study, we consider this effect by adding a space-dependent magnetic field that appears in some applications of semiconductor heterostructures and graphene.[18−19]

The form of the vector potential (2) verifies the Lorentz gauge. From the electromagnetic field tensor the Lorentz invariants are found to be

$$F^{\pi\varrho}F_{\pi\varrho} = 2(B^2 - E^2) = 2(B_0^2\,\mathrm{e}^{-2\nu x} - E_0^2\,\mathrm{e}^{-2\xi t})\,, \quad (3)$$

$$F^{\pi\varrho}F^{*}_{\pi\varrho} = 4\vec{E}\cdot\vec{B} = 4B_0E_0\,\mathrm{e}^{-(\nu x+\xi t)}\,. \quad (4)$$

Then, we infer that electric and magnetic fields will continue simultaneously in all Lorentz frames.

We apply the Bogoulibov transformation technique for calculating the number density of the created particles. The positive and negative frequency states are identified with a quasiclassical method. The approach is that, first exact solution for the Hamilton–Jacobi equation is obtained and it is evaluated for the asymptotic behavior. Then, Dirac equation is solved analytically and the eigenfunctions are computed and compared with the quasiclassical solutions to define the positive and negative frequency states.

2 Solutions of the Hamilton–Jacobi Equation

For the identification of positive and negative frequency modes we follow the method based on the exact solution of relativistic Hamilton–Jacobi (HJ) equation.[7] This method allows to identify the behavior of solutions in the asymptotic regions.

The relativistic HJ equation for the action S is given by:[7]

$$\eta^{\epsilon\theta}\Big[\frac{\partial S}{\partial x^{\epsilon}} - eA_{\epsilon}\Big]\Big[\frac{\partial S}{\partial x^{\theta}} - eA_{\theta}\Big] + m^2 = 0\,, \quad (5)$$

where $\eta^{\epsilon\theta} = (1,-1,-1,-1)$ is the Minkowski metric, the Greek indices ϵ and θ run from 0 to 3, m is the mass of the particle and A_ϵ is the 4-vector electromagnetic potential. Since A_ϵ given by (2) depends on the time and x coordinate of the space, we can separate the solution of the HJ equation as:

$$S(t,\vec{x}\,) = F(x) + Q(t) + (yk_y + zk_z)\,. \quad (6)$$

Here k_y and k_z are conserved momenta resulting from the symmetries of given electromagnetic gauge (2). Substitution of Eq. (6) into Eq. (5) yields

$$\dot{Q}^2 - \acute{F}^2 - \Big(k_z - \frac{eE_0}{\xi}\,\mathrm{e}^{-\xi t}\Big)^2 - \Big(k_y + \frac{eB_0}{\nu}\,\mathrm{e}^{-\nu x}\Big)^2 + m^2 = 0\,, \quad (7)$$

where dot and acute represent derivatives with respect to t and x.

Equation (7) yields two first order differential equations as

$$\dot{Q}^2 - \Big(k_z - \frac{eE_0}{\xi}\,\mathrm{e}^{-\xi t}\Big)^2 = c^2, \quad (8)$$

$$\acute{F}^2 + \Big(k_y + \frac{eB_0}{\nu}\,\mathrm{e}^{-\nu x}\Big)^2 - m^2 = c^2, \quad (9)$$

where c^2 is the constant of separation. By defining the variable $\varrho = \mathrm{e}^{-\xi t}$ Eq. (7) becomes

$$Q(\varrho) = -\frac{1}{\xi^2}\int_0^{\infty}\sqrt{\frac{s_1}{\varrho^2} + \frac{s_2}{\varrho} + (e^2 + E_0^2)}\,\mathrm{d}\varrho\,, \quad (10)$$

where $s_1 = \xi^2(c^2 + k_z^2)$, $s_2 = -2\xi k_z eE_0$.

Since the pair creation is caused by the time-dependent external field the dynamics caused by the space coordinates effect the solutions only by a constant and we obtain the solution of the HJ equation for electromagnetic gauge (2) as follow

$$S(\varrho,\vec{x}) = C(\vec{x}) - \frac{1}{\xi^2}\int_0^{\infty}\sqrt{\frac{s_1}{\varrho^2} + \frac{s_2}{\varrho} + (e^2 + E_0^2)}\,\mathrm{d}\varrho\,, \quad (11)$$

where $C(\vec{x})$ is a constant. The dependence of the solution on time is derived by $\psi \to \mathrm{e}^{\mathrm{i}S}$. Then, the asymptotic behaviour of the relativistic wave function is obtained in the following form

$$\psi_{(\varrho\to\infty)} = \mathrm{e}^{\mathrm{i}S} \to C_1(\vec{x})\,\mathrm{e}^{\mp\mathrm{i}\xi\sqrt{k_z^2+c^2}\ln\varrho}, \quad (12)$$

$$\psi_{(\varrho\to 0)} = \mathrm{e}^{\mathrm{i}S} \to C_2(\vec{x})\,\mathrm{e}^{\pm\mathrm{i}(eE_0\varrho\pm 2\xi k_z\ln\varrho)}. \quad (13)$$

Therefore, the upper and lower signs correspond to the negative and positive-frequency modes.

3 Solutions of the Dirac Equation

The Dirac equation for massive spin-1/2 in external electromagnetic fields is given by[20]

$$[\mathrm{i}\gamma^{\varepsilon}\partial_{\varepsilon} + eA_{\varepsilon}\gamma^{\varepsilon} - m]\psi = 0\,, \quad (14)$$

where γ^{ε} are Dirac matrices given in terms of usual Pauli spin matrices $\vec{\sigma}$ as $\gamma^0 = \left(\begin{smallmatrix} I & 0 \\ 0 & -I\end{smallmatrix}\right)$ and $\vec{\gamma} = \left(\begin{smallmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0\end{smallmatrix}\right)$, m and e are the mass and charge of the electron, and ψ is the four-component spinor.

The Dirac equation results in four coupled differential equations for the spinor. In general, exact analytical solutions of this form of the equation are difficult to obtain, especially for mathematically complicated external fields. To overcome this difficulty, Feynmann and Gell–Mann proposed a two-component form of the Dirac equation for electromagnetic fields as follow[21]

$$[(\vec{P} - e\vec{A}\,)^2 + m^2 - e\vec{\sigma}\cdot(\vec{B} + \mathrm{i}\vec{E}\,)]\phi = (p_0 - eA_0)^2\phi\,, \quad (15)$$

where $\vec{\sigma}$ are usual Pauli matrices and $\phi = \left(\begin{smallmatrix}\phi_1 \\ \phi_2\end{smallmatrix}\right)$ are the solutions of the two-component equation. The four-component spinor can be derived from ϕ as follow

$$\psi = \begin{pmatrix} [\vec{\sigma}\cdot(\vec{P} - e\vec{A}\,) + (p_0 - eA_0) + m]\phi \\ [\vec{\sigma}\cdot(\vec{P} - e\vec{A}\,) + (p_0 - eA_0) - m]\phi \end{pmatrix}. \quad (16)$$

Therefore, in order to obtain exact solutions we follow the two-component formalism and use the electromagnetic gauge that results in parallel electric and magnetic fields. Since the gauge field depends on x coordinate and t, both

P_y and P_z are constants of the motion and solutions can be written in the form

$$\phi = \mathrm{e}^{\mathrm{i}(yk_y+zk_z)}\begin{pmatrix}\chi_1(x)T_1(t)\\ \chi_2(x)T_2(t)\end{pmatrix}. \tag{17}$$

Hence, with the usage of Eqs. (2) and (17), Eq. (15) becomes

$$[\hat{F}(x)+\hat{Q}(t)]\chi_s(x)T_s(t)=0\,, \tag{18}$$

where the spin index s has the ± 1 eigenvalues corresponding to the spinors ϕ_1 and ϕ_2, respectively and operators are defined as

$$\hat{F}(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}+k_y^2+k_z^2+m^2+\frac{e^2B_0^2}{\nu^2}\,\mathrm{e}^{-2\nu x} + eB_0\,\mathrm{e}^{-\nu x}\Big(\frac{2k_y}{\nu}-s\Big)\,, \tag{19}$$

$$\hat{Q}(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}+\frac{e^2E_0^2\,\mathrm{e}^{-2\xi t}}{\xi^2}-eE_0\Big(\frac{2k_z}{\xi}+\mathrm{i}s\Big)\,\mathrm{e}^{-\xi t}. \tag{20}$$

Equation (18) can be separated as follows:

$$[\hat{F}(x)-b^2]\chi_s(x)=0\,, \tag{21}$$

$$[\hat{Q}(t)+b^2]T_s(t)=0\,, \tag{22}$$

where b^2 is the constant of separation.

By introducing the variable $\kappa = 2eB_0\,\mathrm{e}^{-\nu x}/\nu^2$ and defining $\chi_s = \mathrm{e}^{\nu x/2}g_s(\kappa)$, Eq. (21) becomes

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}\kappa^2}+\frac{1}{\kappa^2}\Big[\frac{1}{4}+\frac{b^2-(k_y^2+k_z^2+m^2)}{\nu^2}\Big] + \frac{1}{\kappa}\Big(\frac{s}{2}-\frac{k_y}{\nu}\Big)-\frac{1}{4}\Big\}g_s(\kappa)=0\,. \tag{23}$$

This is the Whittaker equation[22] and solutions are given in terms of the Whittaker functions as

$$g_s(\kappa)=[AW_{\tilde{\lambda},\tilde{\mu}}(\kappa)+BM_{\tilde{\lambda},\tilde{\mu}}(\kappa)]\,, \tag{24}$$

where $\tilde{\mu} = \pm\sqrt{k_y^2+k_z^2+m^2-b^2}/\nu$, $\tilde{\lambda} = (s/2-k_y/\nu)$ and A, B are constants. Conditions for the Whittaker functions must be bounded for the variable is given by[22]

$$\frac{1}{2}+\tilde{\mu}-\tilde{\lambda}=-n\,, \tag{25}$$

and taking the positive value of μ we obtain

$$b=\sqrt{k_z^2+m^2-\nu N(\nu N+2k_y)}\,, \tag{26}$$

where $N=(n+(1-s)/2)$, this expression of b reveals the motion of particle in (y,z) plane is quantized. The reason of taking the positive value for b is the form of Fermi–Dirac thermal distribution, in which the energy term has positive sign.

By defining the comoving time as $\rho = (2\mathrm{i}eE_0/\xi^2)\,\mathrm{e}^{-\xi t}$, Eq. (22) reduces

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}+\frac{1}{\rho^2}\Big(\frac{1}{4}+\frac{b^2}{\xi^2}\Big)+\frac{1}{\rho}\Big(-\frac{s}{2}+\frac{\mathrm{i}k_z}{\xi}\Big)-\frac{1}{4}\Big]f_s(\rho)=0\,, \tag{27}$$

where $T_j(\rho) = \mathrm{e}^{\xi t/2}f(\rho)$. This is the Whittaker equation and solutions are given by[22]

$$f_j(\rho)=[CW_{\lambda,\mu}(\rho)+DM_{\lambda,\mu}(\rho)]\,, \tag{28}$$

where $\mu=\pm\mathrm{i}b/\xi$, $\lambda=-s/2+\mathrm{i}k_z/\xi$ and C, D are constants.

Therefore exact solutions are given by

$$\phi = \mathrm{e}^{\mathrm{i}(yk_y+zk_z)}\,\mathrm{e}^{(\xi t+\nu x)/2}[AW_{\tilde{\lambda},\tilde{\mu}}(\kappa)+BM_{\tilde{\lambda},\tilde{\mu}}(\kappa)] \times[CW_{\lambda,\mu}(\rho)+DM_{\lambda,\mu}(\rho)]\,. \tag{29}$$

Complete components of the spinor can be derived by inserting Eq. (29) into Eq. (17).

4 Bogoliubov Transformation Method

The Bogoliubov transformation method is a technique that associates a canonical commutation relation algebra or a canonical anti-commutation relation algebra into another representation, caused by an isomorphism.[23–24] It is a way of understanding the important topics of the physics such as particle creation mechanism, Hawking radiation and Unruh effect.

In the Minkowskian QFT, eigenfunctions of the field equation, ψ, can be written with the help of the mode solutions as[23]

$$\psi=\sum_n(a_n\tau_n+a_n^\dagger\tau_n^*)=\sum_k(b_k\Upsilon_k+b_k^\dagger\Upsilon_k^*)\,, \tag{30}$$

where we have the relations $(\tau_i,\tau_j)=\delta_{ij}$, $(\tau_i^*,\tau_j^*)=\delta_{ij}$, $(\tau_i,\tau_j^*)=0$ and $(\Upsilon_i,\Upsilon_j)=\delta_{ij}$, $(\Upsilon_i^*,\Upsilon_j^*)=\delta_{ij}$, $(\Upsilon_i,\Upsilon_j^*)=0$ for τ and Υ are mode solutions. The τ and Υ can be expanded in terms of each other.

The creation and annihilation operators $a_n^\dagger, b_k^\dagger$ and a_n, b_k are in correlation by the following expressions

$$a_n=\sum_k(\alpha_{kn}b_k+\beta_{kn}^*b_k^\dagger)\,, \tag{31}$$

$$b_k=\sum_n(\alpha_{kn}^*a_n-\beta_{kn}^*a_n^\dagger)\,, \tag{32}$$

α_{kn} and β_{kn} are Bogoliubov coefficients determined by $\alpha_{ij}=(\Upsilon_i,\tau_j)$, $\beta_{ij}=-(\Upsilon_i,\tau_j^*)$. They are related as

$$\sum_i(\alpha_{ni}\alpha_{ki}^*-\beta_{ni}\beta_{ki}^*)=\delta_{nk}\,, \tag{33}$$

$$\sum_i(\alpha_{ni}\beta_{ki}-\beta_{ni}\alpha_{ki})=0\,. \tag{34}$$

Let $|0_a\rangle$ and $|0_b\rangle$, are two states of vacuum in the Fock space and are related to each particle notion in (30). They are represented for all n and k as

$$|0_a\rangle : a_n|0_a\rangle=0\,, \tag{35}$$

$$|0_b\rangle : b_k|0_b\rangle=0\,. \tag{36}$$

If $|0_b\rangle$ is introduced as the usual vacuum, then $|0_a\rangle$ is regarded as a many-particle state. Therefore, the number of Υ_n-mode particles in the state of $|0_a\rangle$ is

$$\langle 0_a|b_k^\dagger b_k|0_a\rangle=\sum_n|\beta_{kn}|^2. \tag{37}$$

If the $\tau_n(x)$ are defined as positive frequency modes and the $\Upsilon_n(x)$ modes are linear unification of them, then $\beta_{jk}=0$. Then, $b_k|0_b\rangle=0$ and $a_k|0_a\rangle=0$. Hence, τ_j and

Υ_k modes have a common vacuum state. If $\beta_{jk} \neq 0$, then Υ_k contain a combination of positive-τ_k and negative-τ_k^* frequency modes.

5 Particle Production

In order to investigate the particle creation we explore the positive and negative frequency Minkowskian "in" and "out" regions. Since the particle creation is an effect induced by the electric field,[20] the time-dependent components of the wave function will be used in steps of the identification of "in" and "out" vacuum solutions.

Therefore with the help of these solutions we can analyze the modes that behave as positive and negative frequency. We proceed to obtain the asymptotic behavior of the solutions of the Dirac equation in the neighborhood of the time singularities.

The asymptotic behavior of $W_{\lambda,\mu}(\rho)$ for $\rho \to \infty$ is[22]

$$W_{\lambda,\mu}(\rho) \to \mathrm{e}^{-\rho/2}\rho^{\lambda}. \tag{38}$$

Then, as $\rho \to \infty$ the positive and negative frequency modes are

$$f_{\infty}^{+} = A_{\infty}^{+} W_{\lambda,\mu}(\rho)\,, \tag{39}$$

$$f_{\infty}^{-} = [A_{\infty}^{+} W_{\lambda,\mu}(\rho)]^* = A_{\infty}^{+} W_{-\lambda,\mu}(-\rho)\,. \tag{40}$$

Therefore, in the limit of $\rho \to \infty$ the asymptotic behaviour of the solution of Eq. (27) will be as

$$f(\rho) = A_{\infty}^{+} W_{\lambda,\mu}(\rho) + A_{\infty}^{+} W_{-\lambda,\mu}(-\rho)\,, \tag{41}$$

which has the analogous behavior of Eq. (12).

Similarly, the asymptotic behavior of $M_{\lambda,\mu}(\rho)$ as $\rho \to 0$ is[23]

$$M_{\lambda,\mu}(\rho) \to \mathrm{e}^{-\rho/2}\rho^{\mu+1/2}\,, \tag{42}$$

and positive and negative frequency modes as $\rho \to 0$ are

$$f_0^{+}(\rho) = B_0^{+} M_{\lambda,\mu}(\rho)\,, \tag{43}$$

$$f_0^{-}(\rho) = [B_0^{+} M_{\lambda,\mu}(\rho)]^* = B_0^{+}(-1)^{-\mu+1/2} M_{\lambda,-\mu}(\rho)\,, \tag{44}$$

where the coefficients B are reel arbitrary constants. Hence, the asymptotic behavior of the solution of Eq. (27) has the following form for $\rho \to 0$

$$f(\rho) = B_0^{+} M_{\lambda,\mu}(\rho) + B_0^{+}(-1)^{-\mu+1/2} M_{\lambda,-\mu}(\rho)\,, \tag{45}$$

which has the similar behavior of Eq. (13).

On account of obtaining the single particle states for $\rho \to 0$ and $\rho \to \infty$ we can obtain the number density of the created fermionic particles with the help of the Bogoliubov coefficients that are derived by using the positive and negative-frequency solutions. The positive-frequency mode at $\rho \to \infty$ can be written as a linear combination of the positive and negative frequency modes at $\rho \to \infty$ in the form

$$f_{\infty}^{+}(\rho) = \alpha f_0^{+}(\rho) + \beta f_0^{-}(\rho)\,. \tag{46}$$

With the help of below relation given for the Whittaker functions,[22]

$$W_{\lambda,\mu}(\rho) = \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\lambda)} M_{\lambda,\mu}(\rho) + \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\lambda)} M_{\lambda,-\mu}(\rho)\,, \tag{47}$$

we find α and β coefficients as follows:

$$\alpha = \frac{A_{\infty}^{+}}{B_0^{+}} \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\lambda)}\,, \tag{48}$$

$$\beta = \frac{A_{\infty}^{+}}{B_0^{+}} \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\lambda)} (\mathrm{e}^{\mathrm{i}\pi})^{\mu-1/2}. \tag{49}$$

From these expressions we obtain

$$\frac{|\alpha|^2}{|\beta|^2} = \mathrm{e}^{2\pi b/\xi} \frac{|\Gamma[1+\mathrm{i}((b-k_z)/\xi)]|^2}{|\Gamma[1-\mathrm{i}((b+k_z)/\xi)]|^2}\,, \tag{50}$$

for spin-up case.

By using the following formula for Gamma functions[22]

$$|\Gamma(1+\mathrm{i}z)|^2 = \frac{\pi z}{\sinh(\pi z)}\,, \tag{51}$$

and considering the normalization condition of the wave function due to the Fermi–Dirac statistics, $|\alpha|^2 + |\beta|^2 = 1$, the number density of the created particles is obtained as follow

$$N \simeq |\beta|^2 = \frac{1}{1+\mathrm{e}^{2\pi b/\xi}(v \sinh \pi\tilde{v}/\tilde{v}\sinh \pi v)}\,, \tag{52}$$

where $v = ((b-k_z)/\xi)$, $\tilde{v} = (b+k_z)/\xi$ and b is given by Eq. (26).

6 Conclusion

The two-component formalism for the Dirac equation proposed by Feynmann and Gell–Mann is used to study the fermionic particle production in the presence of an exponentially damped time-dependent electric field and a magnetic field depending on inverse square of position. The exact solutions are obtained in terms of the Whittaker functions. We should note that these solutions are also valid for the Klein–Gordon particles in the case of $s = 0$. The "in" and "out" vacuum states are identified with the help of the asymptotic solutions of relativistic HJ equation. They are related by the Bogoliubov coefficients that are used to calculate the particle creation number density.

Particle creation number density given in Eq. (52) is not in Fermi–Dirac thermal form. The production of the particles come on the neighborhood of the time singularities with vanishing momentum along the field and they are accelerated up to the longitudinal momentum k_z. Particles produced by the influence of the weak residual response of the electric field become to have very small momentum k_z. For the limit $k_z \to 0$ number density given in Eq. (52) takes the Fermi–Dirac thermal form

$$N = |\beta|^2 \simeq \frac{1}{1+\mathrm{e}^{2\pi b/\xi}}\,, \tag{53}$$

with a temperature $T = \xi/2\pi$. The electric field does not vanish except for $t \to \infty$. The thermal form of the particle creation spectrum is obtained by the whole damping of the electric field, which is the similar case studied in black holes where the whole horizon creates the particle.[9] The results we obtained by Eqs. (52) and (53) are consistent with the results of Ref. [9] in which a similar electric field was considered alone for the problem. It can be easily seen that for the vanishing magnetic field resulting from Eq. (2), the number density given by Eq. (52) can be reduced to the result of Ref. [9].

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