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ANALYTICAL SOLUTIONS OF A SCALAR PARTICLE IN AN ARBITRARY EXTERNAL MAGNETIC FIELD

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ABSTRACT. In the study, analytical solutions and eigenvalues of a non-relativistic scalar particle in an external magnetic field which is exponentially changing with the space are obtained dealing with Schrödinger equation. For this purpose, Asymptotic Iteration Method (AIM), commonly used over the past decade, is used to tackle the problem. Besides, ladder operators of the system are achieved.

1. INTRODUCTION

By solving the eigenvalue problem in quantum mechanics in physics, we get the information about the system we are interested. Trying to investigate the system in this way may be done either relatively or non-relatively. Schrödinger equation, which is used frequently in quantum mechanics [1, 2], is the energy eigenvalue equation that is conctructed to probe the system non-relatively.

The eigenfunctions (eigenstates or wavefunctions) obtained from the eigenvalue problem give clues about how the system evolves in time. Besides, the energy eigenvalues obtained from this eigenvalue equation are "fingerprints" of the system, so to speak. On the other hand, an important component of the energy eigenvalue problem is the potential energy that represents the interactions which the system is exposed to. Therefore, the eigenstates and eigenvalues obtained from the energy eigenvalue problem are the results that define the system for a given potential energy.

Of course, some of mathematical tools are needed to solve this energy eigenvalue problem in quantum mechanics. In the literature, there are many methods such as Nikiforov-Uvarov (NU) method [3], continuous fraction method (CFM) [4], ... The Asymptotic Iteration Method (AIM) [5] is also used to deal with the energy eigenvalue equation, like the methods mentioned above. The advantage of AIM over the others methods is that it can be used for both analytical and numerical solutions of the energy eigenvalue equation [5, 6, 7].

Based on these motivations, non-relativistic eigenstates and energy eigenvalues of a scalar (spinless) particle which travels under the influence of a space-dependent

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exponentially changing external magnetic field are analyitcally (exact solutions) obtained in this study. Besides, ladder operators of the system are achieved. In the problem in which the Schrödinger equation is addressed, the AIM is used as a mathematical tool.

The study is organized as follows: Section 2 gives brief information about AIM while, in Section 3, analytical solutions and the ladders operators of the system tackled are achieved by using AIM. Finally, Section 4 summarizes the results.

2. Outline of Asymptotic Iteration Method (AIM)

The general solution of a linear differential equation in the form of Eq.2.1 can be obtained as in Eq.2.2, using the AIM in which the details can be found in the Ref.[5]

$$y''(x) = \lambda_0(x)y'(x) + s_0(x)y(x), \tag{2.1}$$

$$y(x) = \exp\left(-\int^{x} \alpha(t)dt\right)\left[C_2 + C_1\int^{x} \exp\left(\int^{t} \left(\lambda_0(t) + 2\alpha(t)\right)d\tau\right)dt\right], \tag{2.2}$$

where C_1 an C_2 are constants. All derivatives of the $\lambda_0(x)$ and $s_0(x)$ functions in Eq.2.1 are available in the defined range of the independent variable x. One can achieve the solution of Eq.2.2 via the asymptotic assumption given below for $\lambda_0(x)$ and $s_0(x)$ functions

$$\frac{s_n(x)}{s_{n-1}(x)} = \frac{\lambda_n(x)}{\lambda_{n-1}(x)} \equiv \alpha(x) \tag{2.3}$$

for sufficiently large values of the n positive integer. Apart from these, the functions $\lambda_0(x)$ and $s_0(x)$ have below given iterative characteristics

$$\lambda_n(x) = \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x), \qquad s_n(x) = s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x) \tag{2.4}$$

For using the method for an eigenvalue problem in quantum mechanics, the energy eigenvalues (E_n) can be reached by means of the following equation obtained by Eq.2.3

$$\delta_n(x,E) \equiv s_n(x,E)\lambda_{n-1}(x,E) - \lambda_n(x,E)s_{n-1}(x,E) = 0. \tag{2.5}$$

If energy eigenvalues can be obtained in an analytical form by using Eq.2.5, such problems are "exact solvable" [7, 8]. Otherwise, the eigenvalues can be obtained numerically [9, 10].

By using a manner similar to that of the eigenvalue, the function generator gine in Eq.2.6 is used to find the eigenfunctions of the system (eigenstates or wavefunction)

$$f_n(x) = C_2 \exp\left(-\int^{x} \frac{s_n(u)}{\lambda_n(u)}du\right) \tag{2.6}$$

3. Analytical Solutions of Schrödinger Equation for the System via AIM

In the system we have dealt with, it is assumed that the scalar particle travels within the magnetic field of $\vec{B}=\beta e^{\eta x}\hat{k}$ derived from a vector potential in the form of $\vec{A}=(0,\frac{\beta}{\eta}e^{\eta x},0)$ in three-dimensional cartesian coordinates where β and η are real constants and $x\in(-\infty,\infty)$.

In natural units (i.e. $\hbar=c=1$), Schrödinger equation for a free particle having a mass of m and an electrical charge of q in an external magnetic field is given as [2]

$$\frac{1}{2m}\left(\vec{p}+q\vec{A}\right)^2\psi(\vec{r})=E\psi(\vec{r}) \tag{3.1}$$

where $p_j=-i\frac{\partial}{\partial x_j}$ (j=1, 2, 3 and $x_1\equiv x, x_2\equiv y, x_3\equiv z$) is the momentum and E is the total energy of the particle, while $\vec{A}$ is the vector potential gives rise to the magnetic field.

A magnetic field such as $\vec{B}=\beta e^{\eta x}\hat{k}$ causes the particle to be exposed to a magnetic force only on the x-axis. Thus, the particle is free on y and z axes. So, the y and z components of the momentum of the particle become constant: $p_y=k_y$ and $p_z=k_z$. Thus, if we choose the space-dependent wavefunction in Eq.3.1 as $\psi(\vec{r})=e^{i(yk_y+zkz)}u(x)$ and put the expessions of the magnetic field and the vector potential into the Eq.3.1, we obtain

$$\left[\frac{d^2}{dx^2}-\frac{q^2\beta^2}{\eta^2}e^{2\eta x}-\frac{2qk_y\beta}{\eta}e^{\eta x}-\varepsilon^2\right]u(x)=0 \tag{3.2}$$

where $\varepsilon^2=k_y^2+k_z^2-2mE$.

If we define a new variable as $\nu=\frac{\sqrt{q\beta}}{\eta}e^{\eta x}$ then choose $u(\nu)=\nu^{\frac{1}{2}}\varphi(\nu)$, Eq.3.2 is yielded as

$$\varphi''(\nu)-\left[1+\frac{\sigma}{\nu}+\frac{\mu}{\nu^2}\right]\varphi(\nu)=0 \tag{3.3}$$

in which $\sigma=\frac{2k_y}{\sqrt{q\beta}}$ and $\mu=\frac{\varepsilon^2}{q\beta}-\frac{1}{4}$. According to the singularity in this equation, if we choose $\varphi(\nu)=\nu^{\gamma+1}e^{-\nu}f(\nu)$, we get the AIM form given in Eq.2.1 as follows

$$f''(\nu)-2\left(1-\frac{\gamma+1}{\nu}\right)f'(\nu)-\left(\frac{2(\gamma+1)+\sigma}{\nu}\right)f(\nu)=0 \tag{3.4}$$

where $\gamma=-\frac{1}{2}\pm\sqrt{\mu+\frac{1}{4}}$. If we compare Eq.3.4 and Eq.2.1, we can start the AIM iterations with

$$\lambda_0(\nu)=2\left(1-\frac{\gamma+1}{\nu}\right),\qquad s_0(\nu)=\left(\frac{2(\gamma+1)+\sigma}{\nu}\right) \tag{3.5}$$

functions.

By using the functions in Eq.3.5, one can achieve $\gamma_0 = -\frac{1}{2}(\sigma+2), \gamma_1 = -\frac{1}{2}(\sigma+4), \gamma_2 = -\frac{1}{2}(\sigma+6)$ and $\gamma_3 = -\frac{1}{2}(\sigma+8)$ according to the first-four AIM iterations. This allow us to generalize the γ as

$$\gamma_n = -\frac{1}{2}\left[2(n+1)+\sigma\right] \tag{3.6}$$

with $n = 0, 1, 2, 3, \ldots$ Using Eq.3.6 and the definitions

$$\gamma = -\frac{1}{2} \pm \sqrt{\mu + \frac{1}{4}}$$
$$\mu = \frac{\varepsilon^2}{q\beta} - \frac{1}{4}$$
$$\varepsilon^2 = k_y^2 + k_z^2 - 2mE$$

we can obtain the energy eigenvalues as follows

$$E_n = \frac{1}{2m}\left[(k_y^2 + k_z^2) - q\beta\left(N + \frac{\sigma}{2}\right)^2\right] \tag{3.7}$$

where $N = n + \frac{1}{2}$.

As for the eigenfunctions of the system (see in Eq.3.4), the function generator given in Eq.2.6 is used with the same $\lambda_0(\nu)$ and $s_0(\nu)$ functions in Eq.3.5 for AIM iterations. One can get

$f_0(\nu) = 1,$
$f_1(\nu) = (\sigma+2)\left\{1 + \frac{(-1)}{-\sigma-2}(2\nu)\right\},$
$f_2(\nu) = (\sigma+3)(\sigma+4)\left\{1 + \frac{(-2)}{-\sigma-4}(2\nu) + \frac{(-2)(-1)}{(-\sigma-4)(-\sigma-3)}\frac{(2\nu)^2}{2}\right\},$
$f_3(\nu) = (\sigma+4)(\sigma+5)(\sigma+6)\left\{1 + \frac{(-3)}{-\sigma-6}(2\nu) + \frac{(-3)(-2)}{(-\sigma-6)(-\sigma-5)}\frac{(2\nu)^2}{2} + \frac{(-3)(-2)(-1)}{(-\sigma-6)(-\sigma-5)(-\sigma-4)}\frac{(2\nu)^3}{6}\right\}$

regarding to the first-four AIM iterations. So, the generalized eigenfunctions of Eq.3.4 is achieved as

$$f_n(\nu) = (n+\sigma+1)_n\left\{\sum_{d=0}^{n}\frac{(-n)_d}{(-\sigma-2n)_d}\frac{(2\nu)^d}{d!}\right\} \tag{3.8}$$

or

$$f_n(\nu) = (n+\sigma+1)_n\, {}_1F_1(-n; -\sigma-2n; 2\nu) \tag{3.9}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ is Pochhammer symbol and ${}_1F_1(a;b;z)$ is the Confluent Hypergeometric Functions [11]. Finally, the overall wavefunctions (eigenfunctions or eigenstates) is yielded as

$$\psi_n(\vec{r}) = N_1 e^{i(yk_y+zk_z)-\nu}\nu^{\gamma+\frac{1}{2}}\, {}_1F_1(-n; -\sigma-2n; 2\nu) \tag{3.10}$$

where N_1 is normalization constant.

3.1. **The Ladder Operators of the System.** The ladder operators enable us to investigate a system having dynamical symmetry via Lie algebraic methods [12, 13]. In order to find the ladder operators of the system, it is necessary to obtain firstly the explicit expression of the normalization constant of the wavefunction. For this purpose, the following relationship [11, 14] is used between Confluent Hypergeometric Functions and Laguerre Polynomials

$$ {}_1F_1(-n;\alpha+1;x)=\frac{n!}{(\alpha+1)_n}L_n^{\alpha}(x) \tag{3.11} $$

One can easily write the wavefunction in Eq.3.10 as

$$ \psi_n(\vec{r})=e^{i(yk_y+zk_z)}u_n(\nu) $$

where

$$ u_n(\nu)=N_n e^{-\nu}\nu^{\gamma+\frac{1}{2}}{}_1F_1(-n;-\sigma-2n;2\nu) \tag{3.12} $$

and N_n is the generalized normalization constant we try to find.

So, using the expression in Eq.3.11, the wavefunction $u_n(\nu)$ can be linked to the Laguerre polynomials as follows

$$ u_n(\nu)=N_n e^{-\nu}\nu^{\gamma+\frac{1}{2}}L_n^{(-\sigma-2n-1)}(2\nu) \tag{3.13} $$

By using the normalization rule of the Laguerre polynomial given as [14]

$$ \int_0^{\infty} e^{-t}t^{a}\left[L_n^{(a)}t\right]^2 dt=\frac{\Gamma(n+a+1)}{n!} $$

one can easily normalize the $u_n(\nu)$ then achieve the N_n normalization constant as follows

$$ N_n=\sqrt{\frac{n!}{(n+2\gamma+1)!}} \tag{3.14} $$

Let's suppose that the ladder operators of the system are in the form of [12, 13]

$$ L_{\pm}=A_{\pm}(\nu)\frac{d}{d\nu}+B_{\pm}(\nu) \tag{3.15} $$

and have the following characteristics

$$ L_{\pm}u_n(\nu)=l_{\pm}u_{n\pm1} \tag{3.16} $$

Using the properties given as [14]

$$ t\frac{d}{dt}L_n^m(t)=nL_n^m(t)-(n+m)L_{n-1}^m(t) $$
$$ -(n+m)L_{n-1}^m(t)=(n+1)L_{n+1}^m(t)+(t-1-2n-m)L_n^m(t) $$

we can following differential equations

$$\left[\nu\frac{d}{d\nu}-\left(n+\gamma-\nu+\frac{1}{2}\right)\right]u_n(\nu)=-\sqrt{n(n+2\gamma+1)}u_{n-1}(\nu) \tag{3.17}$$
$$\left[\nu\frac{d}{d\nu}-\left(\nu-n-\gamma-\frac{3}{2}\right)\right]u_n(\nu)=\sqrt{(n+1)(n+2\gamma+2)}u_{n+1}(\nu)$$

If the differential equations in Eq.3.17 are compared with Eq.3.16, the lowering operator and the l_- constant are got as follow

$$\hat{L}_-=-\nu\frac{d}{d\nu}+\left(n+\gamma-\nu+\frac{1}{2}\right) \tag{3.18}$$
$$l_-=\sqrt{n(n+2\gamma+1)}$$

In a similar manner, the raising operator and the l_+ constant are obtained as

$$\hat{L}_+=\nu\frac{d}{d\nu}-\left(\nu-n-\gamma-\frac{3}{2}\right) \tag{3.19}$$
$$l_+=\sqrt{(n+1)(n+2\gamma+2)}$$

4. Conclusion

In the study, exact energy eigenvalues and eigenfunctions of a Schrödinger particle in a space-dependent external magnetic field changes exponentially. Asymptotic Iteration Method (AIM) that is widely used over the past decade, is used as a mathematical tool to deal with the problem. Furthermore, raising and lowerig operators of the system that enable us to investigate a system having dynamical symmetry via Lie algebraic methods [12, 13] are achieved. As a conclusion of the study, we can say that the AIM gives quite accuracy results.

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