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Unveiling students' explorations of tessellations with Scratch through mathematical aesthetics

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ABSTRACT

This study investigates sixth-grade students' explorations of tessellations with Scratch through the mathematical aesthetics. For this purpose, the study analyzed the students' explorations by considering the roles of mathematical aesthetics identified by Sinclair (2004): generative, evaluative, and motivational role. Six middle-school students were asked to create regular and semi-regular tessellations by using Scratch and interviewed individually about the investigation of tessellations. The findings indicated that the students' generative aesthetic responses to mathematical tasks were most evident in their explorations with Scratch. The students recognized the aesthetic value of mathematical entities while working on tessellations. They never gave up studying on tessellations with Scratch and did not need any external motivation to complete the tasks. Implications for research and practice for teaching are drawn.

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KEYWORDS
Mathematical aesthetics; scratch; tessellations

1. Introduction

Approaches to explore the role of aesthetics in mathematics often centre around mathematical thinking, reasoning, and problem-solving (e.g. Presmeg, 2018; Sinclair, 2004). These approaches tend to focus mainly on how mathematicians solve mathematical problems and how aesthetic values guide them along with their ways. Previous research has shown that the aesthetics is a critical and influencing component for mathematicians in their works, specifically in formulating a theory and developing ideas for mathematical proofs (Brinkman & Sriraman, 2009). Based on mathematicians' works (e.g. problem, theorem, and proof), many general descriptors of aesthetic values are defined such as simple and straightforward (Dreyfus & Eisenberg, 1986); economical, elegant, and visually strong (Whitcombe, 1988); and easily provable, beautiful, and conceptually correct (Raman & Öhman, 2013). From a theoretical point of view, such a perspective, known as *an objective* view, is often mentioned and supported by mathematicians (Sinclair, 2009). This growing interest in aesthetic appreciation by mathematicians acknowledges the factors that underlie aesthetic response to mathematical problems and theorems.

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The conception of aesthetics in mathematics education has evolved, and it has been a prominent focus for the last two decades (Eberle, 2014; Koichu et al., 2017; Sinclair, 2004, 2006, 2009). From the educational perspective, much more research relied on the premises of the *subjective* view, which involves the aesthetic judgments depending on previous experience, social factors, and personal taste (e.g. Koichu et al., 2017; Marmur & Koichu, 2016; Sinclair, 2009). Many researchers have suggested that aesthetic appreciation of the mathematics curriculum at different school levels is valuable (e.g. Dreyfus & Eisenberg, 1986; Sinclair, 2004, 2009). Some researchers have focused on how aesthetic values motivate students to learn mathematics and play a role in learning mathematics (e.g. Chen, 2017; Sinclair, 2004, 2008, 2009, 2011). The role of aesthetics bears on whether students can discover the value of mathematical aesthetics as a part of mathematical practice.

There is ample evidence that aesthetic sense is a critical issue in the learning and doing of mathematics, especially in problem-solving (Presmeg, 2018). According to Dreyfus and Eisenberg (1986), it is possible to learn by students appreciating the aesthetics of mathematical thought when they are involved in problem-solving activities, explicitly discussing more than one solution in the classroom, and giving importance *aha* moments of problem-solving activities. In general, technology as a tool might play an instrumental role in such experiences. Although technology has been seen as a teaching tool for teachers to yield more effective and efficient delivery of instruction, the adaptation of technology as a tool for learners has shifted to focusing on a reorganization and augmentation of cognition (Duffy & Cunningham, 1996). Detailed examination of the technological tools' impact on mathematics learning by Zbiek et al. (2007) showed that cognitive tools yield powerful effects on technical and conceptual mathematical activities. More specifically, several studies in the literature have revealed that Scratch as a visual environment tool supports students to elaborate on the development of mathematical thinking (Rodriguez-Martinez et al., 2020) and to support students' learning of mathematical concepts and relations (Daher et al., 2020). However, far too little attention has been paid to how the use of programming activities such as using Scratch affect students' learning and understanding in specific areas of mathematics (Rodriguez-Martinez et al., 2020). In this study, we sought to understand the students' appreciation of aesthetics when involved in a mathematical activity using Scratch. To do this, we recognize the importance of the roles of aesthetics to understand how students make explorations with Scratch and make decisions to respond. Our study intends to investigate 6th-grade students' explorations of tessellations with Scratch through the roles of mathematical aesthetics. The research question for this study is, therefore, 'How can the students' explorations of tessellations with Scratch be explained by the roles of mathematical aesthetics?'

2. Background to the study

2.1. Tessellations

Tessellation (or tiling) is aesthetic in nature, and tessellation examples have potential in aesthetics experiences (Upitis et al., 1997). In addition to being a common practice in art and daily life, tessellations are considered as a topic in which the geometrical properties of two-dimensional patterns are examined (Callingham, 2004). In general, tessellations are

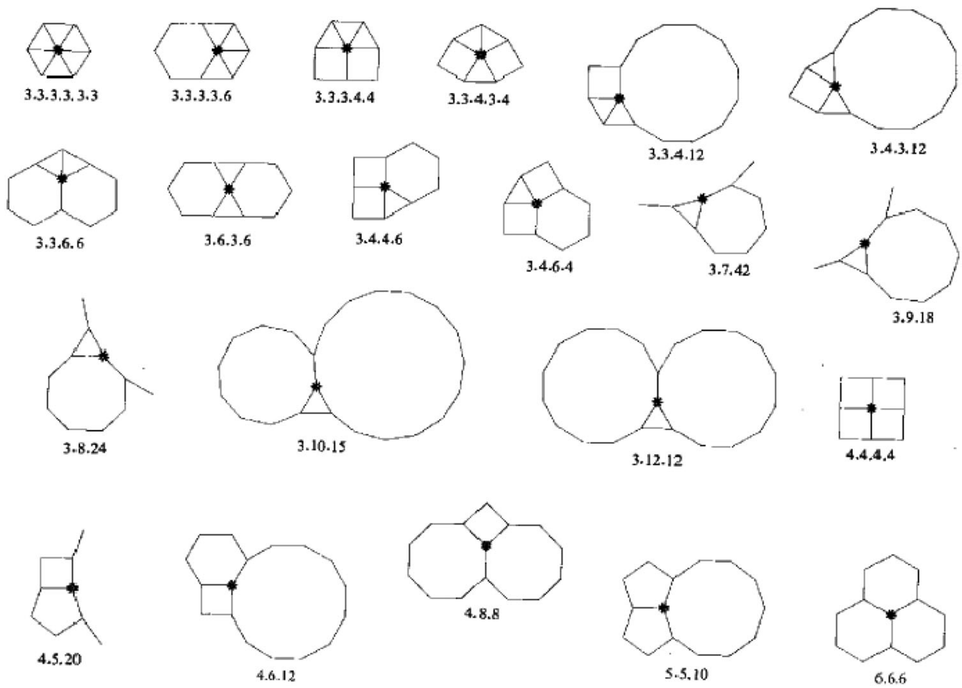


Figure 1. Tessellation combinations (Grünbaum and Shephard, 1987, p.60).

created by filling the shapes without leaving any space on a surface and without overlapping (Willson, 2012). There are three types of tessellations: regular, semi-regular, and irregular (Grünbaum & Shephard, 1987). In particular, regular and semi-regular tessellations consist of regular polygons with the same edge length and angles. Regular tessellations are made up of only one three regular polygons (square, triangle, and hexagon), whereas semi-regular tessellations are formed by more than one different regular polygons. The different types of regular and semi-regular tessellations are represented in Figure 1.

Figure 1 shows 21 different tessellation combinations that could be only formed with regular geometric shapes. For instance, the tessellation code (i.e. 3.4.4.6) shows that it consists of an equilateral triangle, two squares, and a hexagon, respectively. At the vertex (dark points), the sum of the interior angles is $60 + 90 + 90 + 120 = 360$ degrees for this tessellation. Thus, the four shapes fit together with no gaps because the angle measure of the shapes around the point adds up to 360 degrees.

Related research showed that students used the properties of angles and geometrics transformations in their tessellation activities (e.g. Civil, 1995; Eberle, 2015; Kaiser, 1988; Sinclair, 2002). For example, students realized that a tessellation could be created with the regular polygons when the angle completes 360 degrees around a single point (Civil, 1995). Similarly, Kaiser (1988) stated that in the activity of covering a floor with square tiles, students discovered the shapes should not overlap, and there should be no space left between the shapes.

2.2. Role of Scratch in mathematics

The utilization of technology with different perspectives in mathematics education has been discussed in the literature. As Zbiek et al. (2007) pointed out, cognitive tools with trustworthy features and other forms of technologies might assist mathematical learning and teaching serve useful constructs. Foerster (2016) claimed that programming is an integral approach that should be dealt with just like other mathematical tools. As an open-source programming tool, Scratch is used purposefully to promote creativity and enhance the motivation of students to deal with computers (Dohn, 2020). Scratch's programming components, such as expressions, conditions, statements, and variables, are characterized by dragging and dropping blocks (Meerboum-Salant et al., 2013). With the drag and drop method of programming, this environment allows students to design and develop their games, animations, storeys, and simulations, and then share the creations in the online community (Saez-Lopez et al., 2016).

Although programming has not occurred adequately in the scope of mathematics literature (Foerster, 2016), mathematics is one of the subjects using Scratch coding to improve mathematical thinking, positive attitudes towards mathematics, and determined learning outcomes for topics (Dohn, 2020). Much of the current literature on Scratch's role of the improvement in the development of mathematical thinking pays particular attention to the utilization of critical, meta-cognitive, and reflexive skills closely connected to mathematics (Rodriguez-Martinez et al., 2020). Indeed, Scratch as a programming tool might provide students to look for new representations of mathematical ideas and relationships (Hughes et al., 2017). Furthermore, as a functional tool in mathematics (Daher et al., 2020), Scratch provides students with an environment of designing what they think with ready-made code blocks that they sequence to develop a programme (Harvey & Mönig, 2010). Besides Scratch, the other programming environments and tools (e.g. Alice, Karel the Robot, code.org) have an essential role in education since students can develop their mathematical thinking skills with these and similar programmes (Taylor et al., 2010). And build applications and games more simply with the basic and complex programming structures (Utting et al., 2010). With such programmes, students see complex mathematical concepts as engaging activities (Resnick, 2013). As Daher et al. (2020) stated, using Scratch helps students learn about symmetry. In particular, Sinclair (2004) noted that computer-based technology could offer the opportunity to appreciate the importance of the aesthetic dimension in mathematical inquiry.

3. Theoretical framing

The literature from the area of theoretical educational research informed our study. This is reviewed in the scope of mathematical aesthetic in the following sub-section.

3.1. Mathematical aesthetics

Building from work on the mathematical aesthetics (Sinclair, 2004), we aimed to support the aesthetic behaviours in the mathematical inquiry because aesthetic engagement is critically important for considering student learning. To discuss how students develop and internalize the aesthetics of mathematical thought, we need to have a clear idea of what mathematical aesthetics is. In general, mathematical aesthetics is obtained through students' experiences and helps direct them to new and mysterious ideas (Aizikovitsh-Udi, 2014; Eberle, 2011, 2014; Hobbs, 2012). Sinclair (2004) elicited three roles of mathematical aesthetics to understand how students develop an appreciation for mathematical aesthetics: *generative*, *evaluative*, and *motivational*.

3.1.1. Generative role

The term of *generative* describes the mathematicians' generating new ideas, including logical steps (Sinclair, 2006). The generative role of the aesthetics comprises 'nonpropositional modes of reasoning used in the process of inquiry' (Sinclair, 2011, p.7). It guides the decisions and actions made by mathematicians during the process of inquiry to generate mathematical knowledge. Sinclair (2004) identified particular strategies evoking the role of generative aesthetics: *playing*, *establishing intimacy*, and *capitalizing on intuition*. Mathematicians *play* with a mathematical problem by making an exploration, seeking patterns, and looking for a structure; *establish intimacy* with the mathematical object by giving names; and have a particular *intuition* that leads to an unexpected discovery (Sinclair, 2004).

Existing research on the generative role of aesthetics is limited, but several studies present different perspectives to aesthetic responses shaping mathematical knowledge and developing during the mathematical inquiry (Presmeg, 2018; Sinclair & Crespo, 2006; Sinclair, 2006, 2011). These researchers revealed that problem-solving is the most central practice in showing the aesthetic responses during the mathematical inquiry. Sinclair (2006) indicated that the aesthetics could play a generative role in problem-solving procedures. She worked with students and realized how they generated different strategies to solve the problems. For instance, one of the students, who worked for the first time on the Frogs problem in a computer-supported environment, developed a sense of pattern. When the student used symmetry as an effective problem-solving heuristic, she realized that she generated a way for solving the problem. Another student who created squares using the Geometer's Sketchpad dragged the points to different locations and used the tools to construct squares. She figured out how to solve the problem by repeating the procedure. Aesthetics in these examples generate new insights into the students' thought process of solving problems.

With the growing use of digital technologies, the computer provides a 'more tangible, lively, and fun' way for professional and novice mathematicians to generate mathematical understanding and confirm their conjectures (Borwein & Bailey, 2008, p. vii). Sinclair (2011) claimed a similar conclusion about using computers in mathematics challenging and changing the nature of methods used by mathematicians. She highlighted the importance of more research to understand which methods could be valuable and usable for mathematical practice from the point of aesthetical view. By becoming more aware of digital technologies in mathematics education, we can be more explicit about students' applications of mathematical aesthetics and their explorations in mathematics. Additionally, Sinclair (2001) claimed that an applet (i.e. the Colour Calculator) provided a context for analyzing the view of aesthetic in mathematics and generated an aesthetically rich setting for learning mathematics. This applet offered a colourful background and presented options for students to explore patterns in numbers. Students could have an experience in the mathematical discovery and surprise by making explorations with the applet. Therefore, the technology offers strong affordances to make students experience the sense of the mathematical aesthetics (Presmeg, 2018).

Researchers characterized the experience of enlightenment or magical moment as 'aha' moments in mathematics learning and mathematical discovery (Barnes, 2000; Liljedahl, 2005; Satyam, 2016). According to Sinclair (2004), the category of *capitalizing on intuition* relates to this sudden discovery. Moreover, 'aha' moment or experience has a transforming influence on the affective domain of mathematics, resulting in positive beliefs and attitudes towards mathematics (Liljedahl, 2005). Magical moments deal with developing a new perspective on a mathematical concept and understanding an idea with clarity (Barnes, 2000).

3.1.2. Evaluative role

The evaluative role is more recognized and explicit of the three roles since mathematicians mostly used the expressions such as beautiful and elegant for their works (Sinclair, 2006). This role is the objectivist view clarifying the evaluation of characteristics that might contribute to the aesthetic qualities in a solution or proof (Dreyfus & Eisenberg, 1986). It includes the mathematicians' judgment of the great theorems, solutions, and proofs and their decisions about expressing their work (Sinclair, 2004; Sinclair & Crespo, 2006). It is imperative to identify the aesthetic appreciation by mathematicians that can provide information about the evaluative role of the aesthetics since it is the more *reflective* and *after-the-fact* review of the process (Sinclair, 2006).

The discussion around the evaluative role of the aesthetics has revolved almost entirely around mathematicians' aesthetic judgments that played a central role in the process of mathematical thinking. An emerging understanding of the students' evaluative aesthetic responses guides us toward understanding the relation of aesthetics judgments and mathematical knowledge. A body of literature claims students showed somewhat similar appreciation of aesthetics with mathematicians, and they were capable of decisions concerning mathematical entities as mathematicians (Eberle, 2014; Sinclair, 2001). Eberle (2014) found that students showed a similar appreciation of aesthetics with mathematicians when they created and evaluated geometric tessellations. In particular, the use of challenging and unexpected problems has an essential role in predicting students' evaluative responses (Koichu et al., 2017). They indicated that students' evaluative aesthetic response to geometry problems could be successfully predicted and manipulated. Moreover, Brinkman (2009) found, based on a reflective questionnaire associated with the students' evaluative judgments about the beauty of a mathematical problem, that their judgments were related to their problem-solving experiences. For instance, a beautiful problem was easily solved using standard formulas and algorithms for the lower-achieving students. On the other side, for the higher-achievers, a beautiful problem should involve a certain degree of complexity. The students' evaluative judgments about the beauty of a mathematical problem were associated with students' differences in mathematics.

Research showed that mathematicians' views about the beauty of a theorem are related to their emotions and desires and reveal its secret (Pimm & Sinclair, 2006). Mathematical beauty has a great potential in mathematical discovery, especially in selecting appropriate hypotheses to consider (Celluci, 2015). Mamur and Koichu (2016) indicated that students developed a clever solution to a problem after several failed attempts, and they found surprising solutions as beautiful. Theoretically, they showed that there had been a valuable connection between mathematical beauty and surprise. Among different factors associated with the evaluative aesthetic, the role of *surprise* has a special status in the literature on the mathematical aesthetic. *Surprise*, as a kind of emotion, is consisted 'in the individual's response to some event experienced as unexpected, puzzling, or extraordinary' (Declos, 2014, p.3). According to Nunokawa (2001), the surprise is a significant factor in students' appreciation of mathematical ideas; therefore, what expectations they might have in mathematics lessons should be considered.

3.1.3. Motivational role

The motivational role of aesthetics encourages mathematicians to work on a problem (Sinclair, 2006). The aesthetic motivation is essential in mathematical inquiry; however, the motivational aesthetics did not find significant support among mathematicians and is often favoured for framing problems and initial conjectures (Sinclair, 2004). Sinclair (2006, p.97) highlighted the categories of aesthetic motivation that frequently occur in mathematics when mathematicians can be attracted by the 'simplicity, visual appeal, connectedness, and mystery' of the mathematical concepts.

The motivational aesthetics guides students to do mathematics and sustain inquiry (Sinclair, 2004). For instance, Upitis et al. (1997) studied explorations with tessellations to understand how students discover mathematical ideas as mathematicians. They noted that students systematically attempted to create tessellations and found that the sense of fitting tiles well together motivated them. Therefore, the motivational aesthetics can offer potential paths for solving problems and guide students to persist in tackling nonroutine problems (Presmeg, 2018).

4. Method

A *case study* design (Yin, 2013) was applied to examine how students explore tessellations with Scratch. This type of case study is particularly suited to studying when *how* or *why* questions are posed to investigate a current phenomenon within a real-life setting (Yin, 2003). Concerning our study, the case (how the students explore tessellations by using Scratch) provides an insight into the mathematical aesthetics. To obtain a deeper understanding of this topic, we examine students' explorations of tessellations with Scratch through the framework of Sinclair's (2004) roles of mathematical aesthetics. In particular, this study has several reasons for the choice of Scratch, known as a block-based programming tool. One primary reason is in the Turkish curriculum according to which middle-school students would be required to develop an algorithm for solving the problem, test the solution of an algorithm, and examine different algorithms and select the fastest and most accurate solution (Ministry of National Education [MoNE], 2018a). Another reason is that the Tiling question in the sample question set included in the Mathematics Framework draft for the Programme for International Student Assessment (PISA) exam scheduled for 2021 shows how the coding idea can be combined with the tessellation (for details see https://pisa2021-maths.oecd.org/#Examples). This question is designed to assess whether students have geometric representation and computational thinking skills. Therefore, it would be imperative to examine students' explorations of tessellations with a coding programme and present a snapshot of students' experiences about mathematical aesthetics.

4.1. Participants

In this study, six 6th grade students (two girls and four boys), ages 12-13, voluntarily participated from a public school in an urban area of the Mediterranean Region of Turkey. Before this study, based on the conversations with information technology (IT) teachers, these students had already learned the block-based programming logic (e.g. Scratch and Code.org) in the Information Technologies course. Moreover, based on the conversations with mathematics teachers and consistent with mathematics curriculum, 6th-grade students were able to identify and create polygons and recognize their fundamental elements; create triangles according to their angles and sides; and identify and draw the basic elements of the rectangle, parallelogram, rhombus, and trapezoid (MoNE, 2018b). Considering the participants' background, they were ready to explore the tessellations by using Scratch. Pseudonyms of the students (i.e. S1, S2, S3, S4, S5, and S6) were used to keep the names of the participants' confidentiality.

4.2. Procedure

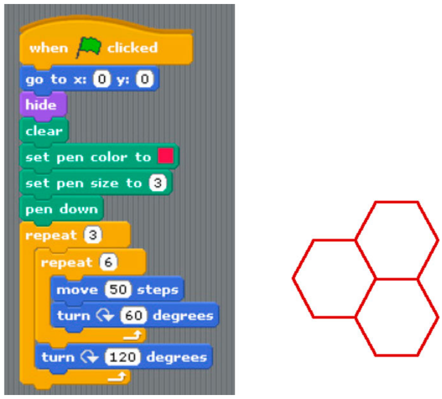
There were three main activities, including different types of tessellations, in this study. In the first activity, students were asked to create regular tessellations with (equilateral) triangle, square, pentagon, hexagon, heptagon, octagon, nonagon, and decagon by using Scratch. They were expected to realize that *regular tessellations* could only be created with triangle, square, and hexagon. In the second activity, they were asked to develop *semi-regular tessellations* with triangles and squares, triangles and hexagons, and squares and hexagons. Finally, they were asked to create *semi-regular tessellations* with triangles, squares, and hexagons in the third activity. If the students did not obtain tessellations, they were asked *why* and elaborate on it.

Students were interviewed individually about the investigation of tessellations by using Scratch. Three semi-structured interviews were conducted with each of the six students to explore their strategies for creating tessellations with Scratch. They spent about two hours 30 minutes in the first activity, two hours 20 minutes in the second activity, and two hours 15 minutes in the third activity. The students were informed that they could leave the activity when they were bored or did not want to complete it. Students were asked to talk aloud as they created regular and semi-regular tessellations in three activities in the individually conducted interviews. Mainly, *think-aloud protocol interviews* (Patton, 2002) were used to reveal what students thought while creating tessellations with Scratch.

4.3. Data analysis

The think-aloud protocol interviews were audiotaped and transcribed; the students' Scratch files for each activity were collected. Each students' transcripts were analyzed based on Sinclair's (2004) three roles of mathematical aesthetics (generative, evaluative, and motivational). Scratch code blocks were also used to provide in-depth information about how students thought and implemented their strategies during the activity. First,

Table 1. Part of data analysis table.

	Student's code blocks, tessellation unit, and reflection
Scratch code blocks and tessellation unit created by Scratch	
Segment of reflection	The exterior angles will be from 120 degrees to 360, and the maximum repetition will be 6. I will go step-by-step. We will turn 60 degrees to the right to add the exterior angles, and then it is essential for me that the inner angles are 120 degrees in the same direction … Yes, let's see now, create a shape, this is a pattern, these places are combined, we can continue it.
Inferences and evidence of aesthetic category	The student found one exterior angle in a regular polygon (360/n), and the interior angle measures of shape add up to 360. The student also tried to cover the plane. She revealed evidence for *playing* according to Sinclair's definition.
Notes	The student identified some rules of regular tessellations with Scratch.

we read and re-read the students' interview transcripts and applied the codes to a short segment of reflections. Then, we coded each segment according to the aesthetic reasons that the students indicated. For the generative role, we used two strategies defined by Sinclair (2004) that seem to help to trigger the generative role: *playing* and *capitalizing on intuition*. In Table 2, the students' strategies coded as *playing* while exploring tessellations using Scratch. When students' attempts suddenly resulted in the discovery, we coded their responses (e.g. aha, it worked) as *capitalizing on intuition*. We coded the students' expressions using aesthetic criteria of *surprise* such as 'very nice motif' and 'wow, it is a nice shape' for the evaluative role. Finally, we created tables to analyze and organize each student's responses for a particular activity. In Table 1, a sample from an analysis of S1 is provided.

The first row contained Scratch code blocks and the tessellation unit created in the first activity. We used the code blocks to interpret how students investigated tessellations and considered the Scratch output to realize what kinds of tessellations they created. The second row contained the related students' reflections while creating tessellations with Scratch, and the third row summarized the authors' descriptions and reasons for aesthetic roles.

Each segment was analyzed based on the aesthetic roles, individually, by the first author, who had master's degree in mathematics education, and second author, who had PhD degree in mathematics education. Next, the third author, who received PhD degree in educational technology, as a third rater performed a reliability check of the analyses. Regarding interrater reliability of analyses, the agreement between the two coders and the external rater was 88 and 92%. The conflicts were resolved after discussing the differences between analyses.

5. Results

We begin with the generative role because the strategies developed by the students were most evident in students' responses and Scratch code blocks. Then, we present the students' expressions under the evaluative role, following the motivational role.

5.1. Generative role

The students' exploration of tessellations with Scratch was examined by considering the aesthetic strategies Sinclair (2004) identified: *playing* and *capitalizing on intuition*. During the activities, the strategies that seemed to trigger the generative aesthetic role and the most crucial part of Scratch code blocks that students used to create tessellations are identified in Table 2.

In the first activity, students were asked to create regular tessellations with triangle, square, pentagon, hexagon, heptagon, octagon, nonagon, and decagon by using Scratch. All students created tessellations and realized that triangles, squares, and regular hexagons are the only regular polygons that will tessellate. The arrangement of Scratch code blocks showed their explorations on looking for appealing tessellation structure. In this activity, students first discovered the rule of *calculating one external angle in a regular polygon* (i.e. 360/n). Figure 2 shows the code blocks students used to create regular tessellations.

Figure 2 shows three different students' code blocks related to finding one exterior angle in hexagon, triangle, and square. For instance, students tried to find the honeycomb

Table 2. Strategies used by students in the activities.

	Strategies Evoking Generative role	Scratch code blocks
Activity I	– Playing with (regular) tessellations by using Scratch code blocks • Finding one exterior angle in a regular polygon (360/n) • Covering an infinite plane • no gaps between shapes • The interior angle measures of shape add up to 360 • the sum of the interior angle surrounding any random vertex point (i.e. 360 degrees) – Capitalizing on intuition	Repeat (numeric) block Turn (360/n) degree
Activity II	– Playing with (semi-regular) tessellations by using Scratch code blocks • Symmetry • Tessellation units – Capitalizing on intuition	Merge repeat blocks: Repeat block Turn (360/n) degree Rotate and reflect Turn (360/n) degree
Activity III	– Playing with tessellations by using Scratch code blocks • Combination of previously developed strategies	Merge repeat blocks Turn (360/n) degree Rotate and reflect

Code blocks for hexagon	Code blocks for triangle	Code blocks for square

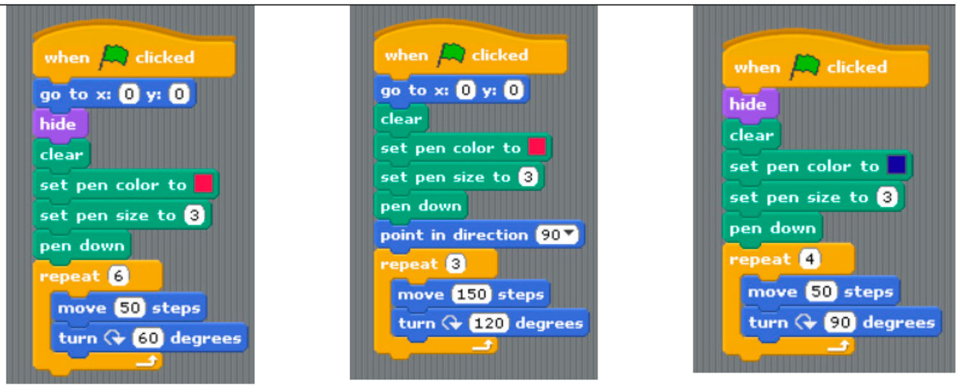


Figure 2. Sample Scratch code blocks.

tessellation by changing the angles and using the code blocks (see the left side of Table 1) to create a hexagon using 60 degrees of rotation. Three students' responses were given below:

> S5: First, I will try 120; if not, I will try 300. 180 or 360 … 120 worked; there is no need to try 300. Ah yes, it worked.
>
> S2: I think the degree is either this or that. I want to change this first. Eeny meeny miny moe um (*children's counting rhyme used to select a person in a game*) which one I should do … Let's try 120. Maybe it will work. Yes, here, it really worked.
>
> S1: I think it will be just right for honeycomb. I will try now. Um, it is not complete. I will add 120 degrees so that it will complete to 180. I think it will work now.

S5 and S2 created the hexagons by trying angles at different attempts, whereas S1 created them distinctly at the first attempt. The students' responses showed that they seemed to be surprised by their attempts. Their responses also related to students' sudden discovery: *Capitalizing on intuition*. It was evident when students felt their decisions were right, especially in the first activity. Moreover, students investigated the rule of regular tessellations in their attempts:

> S6: Their exterior angles will be 360 degrees, not 120 degrees … I will make 'repeat 6' (*talking about Scratch code blocks*). I will turn to the right 60 degrees to sum up the exterior angles, and then it is essential for me that the interior angles are 120 degrees in the same direction … Yeah, let's see … now I have a structure, this is a tessellation. Activity repeats in all directions (*intended to find a pattern that repeats in all directions*). (Tessellations using regular hexagons)
>
> S5: In my opinion, there should not be any gap between the two shapes for tessellations. (Tessellations using regular hexagons)

In these excerpts, students identified the rules of regular tessellations, namely *covering an infinite plane*, *filling gaps between shapes*, and *arrangement polygons surrounding any random vertex point of a tessellation*. In particular, the last strategy was more evident in the following excerpts. When they were asked whether tessellations could be created using pentagon, heptagon, octagon, nonagon, and decagon, they explained that it was not possible to tile planes using these polygons on their own without any spaces or overlaps:

S1: Let's try it. It has eight corners. Let's do it with eight repetitions (*talking about Scratch code blocks*). Yes, it works … Now again, to make it more, do three repetitions. It turns out 135 degrees. And, I think there is no need to try. There are two octagons here. I find 90 degrees … space for a square. This is not a tessellation because this part is too much (*talking about gaps between shapes)* (*Tessellation activity with octagons*)

S4: Now, for example, $90 + 90 + 90 + 90 = 360$, the angles of the squares … Four squares would complete 360 (*talking about the sum of the interior angle measures of four squares surrounding any vertex of a polygon),* but they can't complement each other in this example. 135, 135 is not enough, so I think that's why (*talking about the sum must be 360 degrees*) (*Tessellation activity with octagons*)

In brief, in the first activity, all students created regular tessellations and used several strategies to evoke the generative aesthetics. In the second activity, students used the technique of playing with semi-regular tessellations, including *symmetry* and *tessellation units*. Figure 3 shows an example of how S1 combined different strategies to create tessellations with squares and triangles.

As seen in Figure 2, S1 used the strategy of finding one exterior angle in a regular polygon (360/n) to create a square, applying the 'rotate and translate' movement by using the code block (i.e. point in direction) to create the second square, finding one exterior angle in a regular polygon to create a triangle, and rotating this triangle 60 degrees three times, respectively. In this activity, students noticed *rotational symmetry* by using 90 degrees of rotation of a square which repeats itself with a 'point in direction' code block. The use of rotational symmetry in the activity was evident in the students' statements:

S2: Hey, look! There is a point here. This point goes over many lines, right? The right and bottom are complete, but there it's empty. I can complete it here by making a triangle. Now it is okay. There will be one triangle. I wonder if its direction matters? Let's see … yes, it matters.

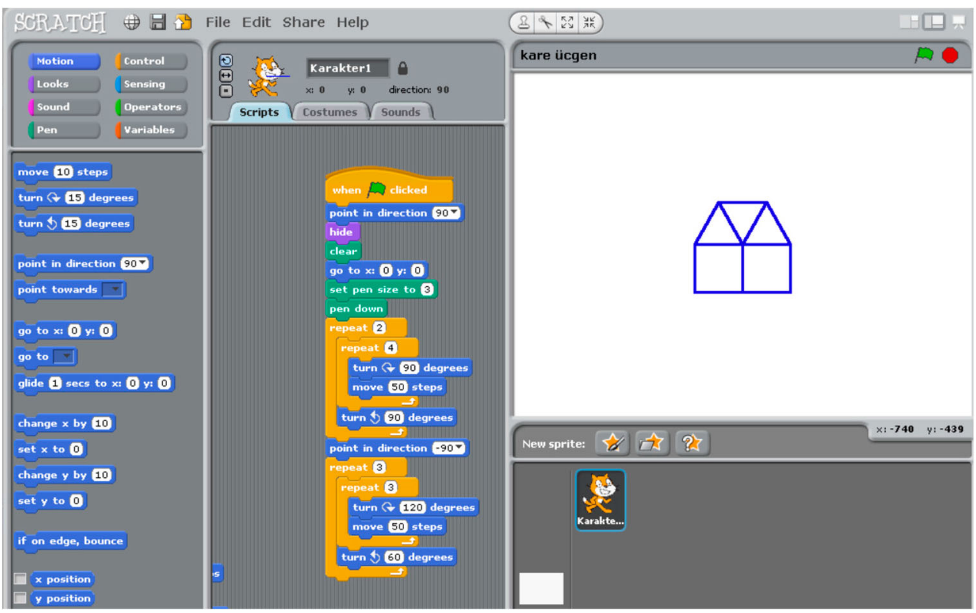


Figure 3. Scratch code blocks created by S1.

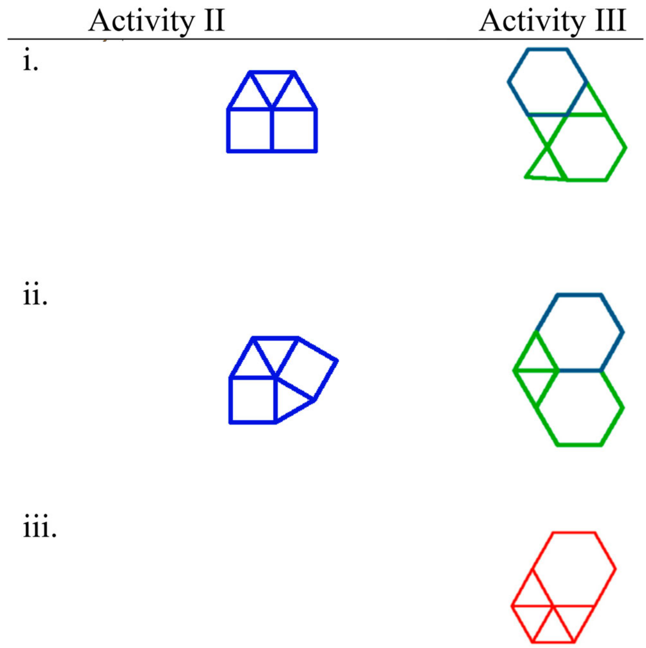


Figure 4. Tessellation Units Created by Students using Scratch.

> Let me try 90 degrees here, let me try −90 degrees, let me try 0, I will do 180 degrees. It doesn't work. What if I give a number myself, um, for example, 300 degrees, it's too big … I think it's not okay, aha I got close. Let me give 360 degrees. It is a frequently used number. Hmm, the two look precisely equal. It is 330 degrees in the middle. I hope it works. Aha, it works; this is a structure (*talking about creating a tessellation unit*)

> *S1*: I have to rotate triangles. I will use the direction to avoid rotating in another direction. Let's try the direction of 90 degrees, and again it is in the same place. Let's try −90 degrees, this time it goes up. 180 degrees yes, I think it will work at 0, but it is inside. This time it goes right. Then we will do it manually. Let me try 150 degrees at first.

S2 and S1 want to reflect triangles using the code block 'point in direction −90 degrees'. Students' expressions such as 'aha it worked' could also be considered an indication of *capitalizing on intuition.* Moreover, the *tessellation unit* was notable in the second and third activities since students investigated different combinations of semi-regular tessellations. In the third activity, students were asked to create tessellations using three polygons (i.e. hexagon, square, and triangle). Figure 4 shows tessellation units created by students using Scratch in the activities.

All students created semi-regular tessellations in the activities. In the third activity, students used the strategies that they investigated in the previous exercises.

5.2. Evaluative role

The evaluative aesthetics was identified in students' decisions about expressing their works. They recognized the aesthetic value of mathematical entities while working on tessellations. *Surprise* included instances when students were surprised by the results of their creating tessellations. Students' expressions such as 'very nice motif, wow it is a nice shape, and very nice' were the most common expressions of aesthetic appreciation referred to tessellations created in the process of their mathematical inquiry. Some students' expressions are in the following:

> *S1:* Here, its edge is too much for here. Let's try here, what can we do, what is missing? Hmm, we will do it again like a location. Let's write 42 to −24. What if I put one? Actually, it comes to my mind. I can make it by putting 3 in here, and this turned sideways. It rotates at 120 degrees. If we do this 2 it will be ok. I think it is very nice. (*Semi-regular tessellation activity with square and triangle*)

> *S2:* I got it. I think it is 72 degrees. Let me bring the cat into the center. I will use this to bring the cat. As you can see, it looks in excellent shape. I will try 72 degrees again. I hope it works now. Aha, it worked. But wows it is in perfect shape. (*Regular tessellation activity with pentagon*)

> *S3:* I have made a mistake somewhere. Let me think, what if I try same like this. It draws next to it. Hmmm, I will change the place of these 90 degrees. It looks like an umbrella. It becomes a very nice shape. (*Semi-regular tessellation activity with square and triangle*)

> *S6:* It is repeating here. I think there is a problem. Let's try it now. Yes, it has become an adorable shape. I wouldn't think it would be like this. This is a motif, a lovely motif. (*Regular tessellation activity with square*)

As can be seen from the students' expressions, they were very impressed that they created tessellations. It seemed that surprise arose as they found tessellations. Since we did not ask students to evaluate their tessellations, students did not make any further explanations. Moreover, there was some evidence that *real-world connections* played aesthetic role in creating tessellations. The reasonably rare instances of real-world connections generated excitement among the students, as in the umbrella example of S3.

5.3. Motivational role

Each student studied on tessellations by using Scratch over two hours in each activity. Students did not need any external motivation to complete the tessellation tasks and were motivated by their satisfaction from creating tessellations. They stayed incredibly motivated during all three activities and achieved the tasks. It seemed that the motivational aesthetics had great potential for students' mathematical inquiry.

6. Discussion

The prior research demonstrated the importance of the role of mathematical aesthetics in students' mathematical understanding (Eberle, 2014; Sinclair, 2004); however, it offers little explanation for the programming tool in general and Scratch, in particular. Therefore, the present study examined a group of middle-school students' explorations with Scratch through the role of mathematical aesthetics (i.e. generative, evaluative, and motivational) (Sinclair, 2004). Furthermore, since these roles play a part in doing mathematics and motivating and sustaining inquiry (Sinclair, 2004), we attempted to display that they helped probe students' aesthetic appreciation and mathematical exploration through Scratch.

Regarding the generative role of the aesthetics, there are different strategies (i.e. *playing*, *establishing intimacy,* and *capitalizing on intuition*) supporting mathematicians and students as an inquirer generate knowledge and gain perspective about the relations in the mathematical structure (Sinclair, 2006). In this study, in terms of the *playing* strategy, the findings showed that the students experienced a strong sense of fitting regular shapes together surrounding any random vertex point of a tessellation and discovered the rules of tessellations by using Scratch code blocks. Remarkably, they entered into the tessellation tasks and find the rule of calculating one external angle in a regular polygon (i.e. 360/n) by manipulating the code blocks. This strategy at once suggested and confirmed by the students played a *generative* role in coming to a sense of view about tessellations. This kind of experience in tessellations was noted as *aesthetic* in nature by Upitis et al. (1997) in a study of students developing a description of the characteristics of tessellations. Moreover, through the manipulation of code blocks, Scratch provided a way to engage the students better to generate aesthetic responses. In similar findings to Harvey & Mönig, (2010), Scratch provided the students with an environment of designing create tessellations that they think with ready-made code blocks in their activities.

In a similar vein, the students' sudden discovery after several attempts to explore tessellations were related to working with *intuition*. With intuition, the students described their decisions as being founded on a specific form of insight rather than reasoning. It could be argued that a solution suddenly became clear while the students were working on the tasks in the activities. The 'aha' experience in problem-solving as a breakthrough might describe the students' sudden clarification of the solution and an instance of appreciation of mathematical aesthetics. This is in line with Sinclair's (2004) examples of *capitalizing on intuition* about the strategies in the generative role of the aesthetics. Satyam (2016) underlined that if there are no eureka or 'aha' moments in mathematics, students would not have any opportunities to experience surprise. Findings of the study indicated that surprise arose as the students explored the tessellations in the first and second acts, in which the 'aha' moments also occurred. Indeed, Eberle (2014) highlighted that when students engaged in mathematical inquiry, surprise as an aesthetic criterion played a significant role in the generative role rather than evaluative one. In parallel to this discussion of surprise, we realized that surprise could not be only expressed as an evaluative aesthetics since it constantly occurred in the 'after-the-fact evaluative role of the aesthetic' as stated by Sinclair (2006, p.134) but also could be described as generative aesthetic because of the strong relationship with the 'aha' moments.

Regarding the evaluative role of the aesthetics, the students were pleased with having created a tessellation by discovering tessellations, in which the aesthetics had an *evaluative* function. According to Sinclair (2006), the generative role of the aesthetics is related to the discovery stage, whereas the evaluative one belongs to the justification stage of the mathematical process. Therefore, the identified two roles, namely generative and evaluative, played a pivotal role in students' discovering and justifying the rule of creating tessellations with Scratch. Sinclair (2006) also mentioned that the computer-supported environment, due to its visual and experimental nature, helps students explore their methods to solve the problems. Remarkably, the visualization provided by Scratch might bring a different perspective to mathematical aesthetics. The findings of this study showed that the facilities provided by Scratch make aesthetically pleasing representations. This is in line with Presmeg's (2018) recommendations that the problem-solving process in a web project was pleasurable and exciting for students. Besides, computer programmes (e.g.

Mapple, Cinderella, The Geometer's Sketchpad) bring a different perspective to the classical solutions of the very old theorems known, making the evidence visually elegant and consequently changing the aesthetic perspective (Borwein, 2008). According to Jackiw (2006), in dynamic geometry programmes such as the Geometer's Sketchpad, magical movements (i.e. dragging) show its connection with aesthetics, such as dynamic environments (Mouse in our hand, cursor on the screen) work like a prosthesis of our desire and imagination. In this study, while students were performing rotations and transformations with the code blocks, they were able to follow the movements and positions of the geometric shapes formed in the section showing the movement of the shapes in Scratch at the same time and noticed that the shapes come together with a particular order and rule in the process of creating a tessellation.

Considering the idea that Scratch helps students improve their mathematical thinking skills (Resnick, 2013; Taylor et al., 2010) and should be a standard tool in mathematics in school (Foerster, 2016), in this study, the students had the opportunity to quickly try and improve the strategies they developed with the fast and mobile structure of the code blocks. Furthermore, Daher et al. (2020) demonstrated that learning symmetry concepts and relations might be achieved more easily with Scratch as a functional tool. Therefore, future research might consider studying how Scratch supports the students' mathematical aesthetics and thinking. This aspect might include their engagement in mathematical thinking and their aesthetic appreciations to explore mathematical concepts.

Regarding the motivational role of the aesthetics, the students never gave up studying on tessellations for hours at a time and did not need any external motivation to complete the tasks. Students' intrinsic motivation of completing the tasks might allow students to generate ways of creating tessellations. As demonstrated in prior research (Upitis et al., 1997), the feeling of putting tessellations together motivated the students. Still, there has been a strong need to emphasize the significance of the students' actions and attempts to display the motivational aesthetics. Our study also indicated that the tessellation tasks the students worked on seemed particularly motivational in nature and included non-routine procedures. In the face of difficulty in solving non-routine problems, the motivational aesthetics has the potential to sustain persistence (Presmeg, 2018). Therefore, we could claim that creating opportunities for students to develop a sense of mathematical aesthetics within non-routine problems and programming environments might be possible. Besides, Sinclair (2004) specified that problem selection has a very central position in the motivational aesthetics. Nonetheless, there has been a great need to clarify which issues are likely to draw students' attention in such environments. By integrating problem selection into the investigation, future researchers may consider the motivational aesthetic for providing an exhaustive perspective to the mathematical aesthetics.

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