Notes on Intuitionistic Fuzzy Sets
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Deferred statistical convergence of sequences in intuitionistic fuzzy normed spaces

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Abstract: In this paper, the intuitionistic fuzzy deferred statistical convergence in the intuitionistic fuzzy normed space is defined by considering deferred density given in [13]. Besides the main properties of this new method, it is compared with intuitionistic fuzzy statistical convergence and itself under different restrictions on the method. Some special cases of the obtained results are coincided with known results in literature.

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1 Introduction and definitions

Fuzzy set theory began with the work of Zadeh [21] in 1965 as an alternative approach to the decision making problems in engineering. Since then, many researchers have been interested in this new subject and many of them have tried to establish whether analogues of classical theories are true or not in the fuzzy case.

In the last two decades, fuzzy logic finds application in different areas of science such as nonlinear dynamic system [10], control of chaos [8], quantum physics [14], etc. It has also many applications in different brances of mathematics; metric and topological spaces ([2, 6, 9, 11]) and approximation theory [18], etc.

The notation of statistical convergence of real valued sequences is first defined by Fast and Steinhaus in 1951 [7] and [20].

Let K be a subset of the set of positive natural numbers $\mathbb{N}$ and let $K(n)$ denote the set $\{k \leq n,\ k \in K\} = [0, n] \cap K$. Asymptotic (or natural) density of the subset K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |K(n)|$$

(if this limit exists), where $|K(n)|$ denotes the cardinality of the set $K(n)$.

Definition 1. *[7] A number sequence $x = (x_k)$ is said to be statistically convergent to the number l if for each $\varepsilon > 0$ we have*

$$\delta(K) = \lim_{n} \frac{1}{n} |\{k \leq n : k \in K\}|.$$

Remark 1. *We remark that every convergent sequence is statistically $(l(S))$ convergent, but not conversely.*

In 1932, R. P. Agnew [1] defined the deferred Cesaro mean $D_{p,q}$ of a sequence $x = (x_n)$ by

$$(D_{p,q}x)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k,$$

where $\{p(n)\}_n$ and $\{q(n)\}_n$ are sequences of positive natural numbers under which

$$p(n) < q(n) \text{ and } q(n) \longrightarrow \infty. \tag{1}$$

Definition 2. *A sequence $x = (x_n)$ is called*

1. *deferred Cesaro convergence to L if*

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} (x_k - L) \longrightarrow 0;$$

2. *strongly deferred Cesaro convergence to L if*

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - L| \longrightarrow 0.$$

We denote $\lim_{n\to\infty} x_n = l(DS[p, q])$.

Let K be an arbitrary subset of $\mathbb{N}$ and

$$K_{p,q}(n) = \{p(n) < k \leq q(n),\ k \in K\}$$

be an associated set of K for the arbitrary sequences $p(n)$ and $q(n)$ satisfying (1).

Definition 3. [13] *Let K be an arbitrary subset of $\mathbb{N}$. If the following limit*

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{q(n) - p(n)} |K_{p,q}(n)|$$

exists, then $\delta_{p,q}(K)$ is called deferred density of the subset K.

Definition 4. *A sequence $x = (x_k)$ is said to be deferred statistically convergent to $l \in \mathbb{R}$ if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} |\{k, p(n) < k \leq q(n) \ : \ |x_k - l| \geq \varepsilon\}| = 0.$$

It is denoted by $\lim_{n\to\infty} x_n = l(D[p,q])$.

The theory of intuitionistic fuzzy sets was introduced by Atanassov in [3, 4] as a generalization of fuzzy sets theory, and it has been extensively used in decision-making problems. The concept of intuitionistic fuzzy metric space was introduced in [18]. Also, in [18] and [15], definition of intuitionistic fuzzy normed space has been given.

Definition 5. [19] *A triangular norm (t-norm) is a continuous mapping $* : [0,1] \times [0,1] \longrightarrow [0,1]$, such that $(S, *)$ is an Abelian monoid with unit one and $c * d \leq a * b$ if $c \leq a$ and $d \leq b$, for all a, b, c, $d \in [0,1]$.*

Definition 6. [19] *A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$, is said to be a continuous t-conorm if it satisfies the following conditions:*

1. *$\diamond$ is associative and commutative,*
2. *$\diamond$ is continuous,*
3. *$a \diamond 0 = a$, for all $a \in [0,1]$,*
4. *$c \diamond d \leq a \diamond b$ if $c \leq a$ and $d \leq b$, for all a, b, c, $d \in [0,1]$.*

Using the continuous t-norm and t-conorm, Saadati and Park [18] have recently introduced the concept of intuitionistic fuzzy normed space, as follows.

Definition 7. *The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ, ν are fuzzy sets on $X \times (0,1)$ satisfying the following conditions for every x, $y \in X$, and s, $t > 0$:*

1. *$\mu(x,t) + \nu(x,t) \leq 1$,*
2. *$\mu(x,t) > 0$,*
3. *$\mu(x,t) = 1$ if and only if $x = 0$,*
4. *$\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$, for each $\alpha \neq 0$,*
5. *$\mu(x,t) * \mu(y,s) \leq \mu(x+y, s+t)$,*

6. $\mu(x, .) : (0, \infty) \longrightarrow [0, 1]$ *is continuous,*

7. $\lim_{t\to\infty} \mu(x, t) = 1$ *and* $\lim_{t\to 0} \mu(x, t) = 0$*,*

8. $\nu(x, t) < 1$*,*

9. $\nu(\alpha x, t) = \nu(x, \frac{t}{\alpha})$*, for each* $\alpha \neq 0$*,*

10. $\nu(x, t) \diamond \nu(y, s) \geq \nu(x + y, s + t)$*,*

11. $\lim_{t\to\infty} \nu(x, t) = 0$ *and* $\lim_{t\to 0} \nu(x, t) = 1$*.*

In this case (μ, ν) *is called an intuitionistic fuzzy norm.*

Example. Let $(X, \|.\|)$ be a normed space. If we take $a * b := a \wedge b$, $a \diamond b := a \vee b$ and $\mu_0(x, t) := \dfrac{t}{t + \|x\|}$ and $\nu_0(x, t) := \dfrac{\|x\|}{t + \|x\|}$ for every $x \in X$. So, $(X, \mu_0, \nu_0, *, \diamond)$ is an IFNS.

Definition 8. *Let* $(X, \mu, \nu, *, \diamond)$ *be an IFNS. Then, a sequence* $x = (x_k)$ *is said to be convergent to* $l \in X$ *with respect to the intuitionistic fuzzy norm* (μ, ν) *if for all* $\varepsilon \in (0, 1)$ *and* $t > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$\mu(x_k - l, t) > 1 - \varepsilon \text{ and } \nu(x_k - l, t) < \varepsilon \tag{2}$$

hold for all $k \geq k_0$*. We denote* $(\mu, \nu) - \lim x = l$ *or* $x_k \overset{(\mu,\nu)}{\longrightarrow} l$

Let us denote $1 - \mu(x, t)$ by $\tilde{\mu}(x, t)$. Hence, the first part of (2) in Definition 8 can be restated as follows:

$$\tilde{\mu}(x_k - l, t) < \varepsilon$$

We are going to use this notation in the proofs of some theorems only for simplicity.

Definition 9. *Let* $(X, \mu, \nu, *, \diamond)$ *be an IFNS and* (x_k) *be a sequence of* X*. It is said that* (x_k) *is a Cauchy sequence if for all* $\varepsilon \in (0, 1)$ *and* $t > 0$ *there is* $n_0 \in \mathbb{N}$*, such that* $\mu(x_k - x_n, t) > 1 - \varepsilon$ *and* $\nu(x_k - x_n, t) < \varepsilon$ *hold for all* k*,* $n \geq n_0$*.*

Definition 10. *Let* $(X, \mu, \nu, *, \diamond)$ *be an IFNS and* (x_k) *be a sequence of* X*. It is said that* (x_k) *is a bounded sequence if there exists a positive real number* M *such that* $\mu(x_k, t) > 1 - M$ *and* $\nu(x_k, t) < M$ *hold for all* $k \in \mathbb{N}$*.*

Definition 11. *Let* $(X, \mu, \nu, *, \diamond)$ *be an IFSN. Then a sequence* $x = (x_k)$ *is said to be statistically convergent to* $l \in X$ *with respect to the intuitionistic fuzzy norm* (μ, ν) *if every* $\varepsilon \in (0, 1)$ *and* $t > 0$*,*

$$\delta(k \in \mathbb{N} : \mu(x_k - l, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon) = 0.$$

Or equivently

$$\delta(k \in \mathbb{N} : \mu(x_k - l, t) > 1 - \varepsilon \text{ or } \nu(x_k - l, t) < \varepsilon) = 1.$$

In this case, we write

$$x_k \to l(S(\mu, \nu)).$$

In the paper [12], the statistical convergence of sequences in IFNS is studied. Later in [16, 17], lacunary statistical convergence and λ-statistical convergence of sequences in IFNS are defined and some interesting results are given, respectively.

The main aim of this paper is to define deferred statistical convergence of sequence in IFNS and to give a generalized version of the results from [12, 17, 16] and some others.

Definition 12. *Let $(X, \mu, \nu, *, \diamond)$ be an IFSN and $x = (x_k)$ be a sequence of X. Then the sequence $x = (x_k)$ is said to be $D_{p,q}$-convergent to l with respect to the intuitionistic fuzzy norm (μ, ν) and is denoted by $x_k \to l(S(\mu, \nu))$ if for every $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \mu\left(x_k - l, t\right) > 1 - \varepsilon \text{ and } \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \nu\left(x_k - l, t\right) < \varepsilon$$

hold for all $n > n_0$.

Definition 13. *Let $(X, \mu, \nu, *, \diamond)$ be an IFSN and (x_k) be a sequence of X. Then, the sequence $x = (x_k)$ is said to be deferred statistically convergent to l, if for all $\varepsilon \in (0, 1)$ and $t > 0$,*

$$\delta_p^q(k \in \mathbb{N} : \mu(x_k - l, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon) = 0$$

Or equivently

$$\delta_p^q(k \in \mathbb{N} : \mu(x_k - l, t) > 1 - \varepsilon \text{ or } \nu(x_k - l, t) < \varepsilon) = 1$$

In this case, we write

$$x_k \to l(D_p^q S(\mu, \nu)).$$

Remark 2. *It is clear that,*

(a) If $q(n) = n$ and $p(n) = n - 1$, then Definition 13 coincides with the Definition 9,

(b) If $q(n) = n$ and $p(n) = 0$, then Definition 13 coincides with the Definition 11 given in [12],

(c) If $q(n) = k_n$ and $p(n) = k_{n-1}$ (for any lacunary sequence of non-negative integers with $k_n - k_{n-1} \to \infty$ as $n \to \infty$), then Definition 13 turns to lacunary statistical convergence in IFNS, given in [17],

(d) If $q(n) = \lambda_n$ and $p(n) = n - \lambda_n$ (where λ_n is a non-decreasing sequence of positive natural numbers denting to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 0$), then Definition 13 coincides with the definition in [16],

(e) If $q(n) = \lambda_n$ and $p(n) = 0$ (where λ_n is a strictly increasing sequence of positive natural numbers), then Definition 13 turned to λ-statistical convergence in IFNS (But this type of convergence in IFNS has not been investigated until today).

2 $D_p^q S(\mu,\nu)$-convergence in IFNS

Some work related to the convergence of sequences in several normed linear spaces in a fuzzy setting can be found in [5]. In this part, we are going to give the main results about $D_p^q S(\mu,\nu)$.

Theorem 1. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS, (x_k) be a sequence of X. Then, $x_k \to l(\mu,\nu)$, implies $x_k \xrightarrow{(\mu,\nu)} l(D_p^q(\mu,\nu))$.*

Proof. For every $\varepsilon \in (0,1)$ and $t>0$, there exists a number $k_0 \in \mathbb{N}$ such that

$$\mu(x_k - l, t) > 1-\varepsilon \text{ and } \nu(x_k - l, t) < \varepsilon \tag{3}$$

hold for all $k \geq k_0$. If the inequalities in (3) are assumed from $p(n)+1$ to $q(n)$, then the following inequality is obtained:

$$\sum_{p(n)+1}^{q(n)} \mu(x_k - l, t) > (1-\varepsilon)(q(n)-p(n)) \text{ and } \sum_{p(n)+1}^{q(n)} \nu(x_k - l, t) < \varepsilon(q(n)-p(n)). \tag{4}$$

If both sides of (4) are divided by $(q(n)-p(n))$, then the desired result is obtained. □

Theorem 2. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS, (x_k) be a sequence of X. Then, $x_k \to l(\mu,\nu)$ implies $x_k \xrightarrow{(\mu,\nu)} l(D_p^q S(\mu,\nu))$.*

Proof. By hypothesis, for every $\varepsilon \in (0,1)$ and $t>0$ there exists a number $k_0 \in \mathbb{N}$ such that $\mu(x_k - l, t) > 1-\varepsilon$ and $\nu(x_k - l, t) < \varepsilon$ hold for all $k \geq k_0$.

This guarantees that the cardinality of the set

$$\{k \in \mathbb{N} \,:\, \mu(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon\}$$

is finite. So, immediately we see that

$$\delta_p^q(\{k \in \mathbb{N} \,:\, \mu(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon\}) = 0$$

This gives the proof. □

Remark 3. *The converse of Theorem 2 is not true, in general.*

In this case, let us consider $(\mathbb{R},\mu_0,\nu_0,*,\diamond)$ and

$$x_k := \begin{cases} 1, & k = m^2, \ m \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

For any $\varepsilon>0$ and $t>0$, consider the following set:

$$K_p^q(\varepsilon,t) = \{p(n)+1 \leq k \leq q(n) \,:\, \mu_0(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu_0(x_k - l, t) \geq \varepsilon\}.$$

It is clear that

$$K_p^q(\varepsilon,t) = \left\{p(n)+1 \leq k \leq q(n) \ \ |x_k| \geq \frac{\varepsilon t}{1-\varepsilon}\right\}$$
$$= \{p(n)+1 \leq k \leq q(n) \ : \ k = m^2, \ m \in \mathbb{N}\}$$

and we have

$$\delta_p^q(K_p^q(\varepsilon,t)) \leq \lim_{n\to\infty} \frac{\sqrt{q(n)} - \sqrt{p(n)}}{q(n)-p(n)} = 0.$$

The last inequality gives that $x_n \to 0(D_p^q S(\mu_0,\nu_0))$. But by Lemma 4.10 in [18], the sequence (x_n) is not (μ_0,ν_0) convergent to zero because it is not convergent to zero in $(\mathbb{R}, |\ .\ |)$.

From Definition 13, we can give following results without proof:

Lemma 1. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS. Then, for all $\varepsilon \in (0,1)$ and $t > 0$, the following statements are equivalent:*

(i) $x_k \stackrel{(\mu,\nu)}{\longrightarrow} l(D_p^q S(\mu,\nu))$,

(ii) $\delta_p^q(\{k \in \mathbb{N} \ : \ \mu(x_k - l,t) \leq 1-\varepsilon\}) = \delta_p^q(\{k \in \mathbb{N} \ : \ \nu(x_k - l,t) \geq \varepsilon\}) = 0$,

(iii) $\delta_p^q(\{k \in \mathbb{N} \ : \ \tilde{\mu}(x_k - l,t) \geq \varepsilon\}) = \delta_p^q(\{k \in \mathbb{N} \ : \ \nu(x_k - l,t) \geq \varepsilon\}) = 0$,

(iv) $\delta_p^q(\{k \in \mathbb{N} \ : \ \tilde{\mu}(x_k - l,t) < \varepsilon\}) = \delta_p^q(\{k \in \mathbb{N} \ : \ \nu(x_k - l,t) < \varepsilon\}) = 1$,

(v) $\tilde{\mu}(x_k - l,t) \to 0(D_p^q S)$ *and* $\nu(x_k - l,t) \to 0(D_p^q S)$.

Theorem 3. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS, (x_k) be a sequence of X. Then, the $D_p^q S(\mu,\nu)$ limit of (x_n) is unique.*

Proof. Let us assume that $x_n \to l_1(D_p^q S(\mu,\nu))$ and $x_n \to l_2(D_p^q S(\mu,\nu))$. For an arbitrary $\varepsilon > 0$, let us choose $r < 0$ such that $(1-r)*(1-r) > 1-\varepsilon$ and $r \diamond r < \varepsilon$. Then, from the assumption for every $t > 0$, we have

$$\delta_p^q(K_{\mu,1}(\varepsilon,t)) = \delta_p^q(K_{\nu,1}(\varepsilon,t)) = 0 \text{ and } \delta_p^q(K_{\mu,2}(\varepsilon,t)) = \delta_p^q(K_{\nu,2}(\varepsilon,t)) = 0,$$

where

$$K_{\mu,1}(\varepsilon,t) := \{p(n) < k \leq q(n) \ : \ \mu(x_k - l_1,t) \leq 1-\varepsilon\},$$
$$K_{\mu,2}(\varepsilon,t) := \{p(n) < k \leq q(n) \ : \ \mu(x_k - l_2,t) \leq 1-\varepsilon\}$$

and

$$K_{\nu,1}(\varepsilon,t) := \{p(n) < k \leq q(n) \ : \ \nu(x_k - l_1,t) \geq \varepsilon\},$$
$$K_{\nu,2}(\varepsilon,t) := \{p(n) < k \leq q(n) \ : \ \nu(x_k - l_2,t) \geq \varepsilon\}.$$

If we denote the set

$$K_{\mu,\nu}(\varepsilon,t) = \{K_{\mu,1}(\varepsilon,t) \cup K_{\mu,2}(\varepsilon,t)\} \cap \{K_{\nu,1}(\varepsilon,t) \cup K_{\nu,2}(\varepsilon,t)\},$$

then it is enough to show that $\delta_p^q(K_{\mu,\nu}(\varepsilon,t))=0$, which implies that $\delta_p^q(\mathbb{N}-K_{\mu,\nu}(\varepsilon,t))=1$.

Now, let $k\in\mathbb{N}-K_{\mu,\nu}(\varepsilon,t)$, then there are two cases:

$$k\in\mathbb{N}-\{K_{\mu,1}(\varepsilon,t)\cup K_{\mu,2}(\varepsilon,t)\} \text{ or } k\in\mathbb{N}-\{K_{\nu,1}(\varepsilon,t)\cup K_{\nu,2}(\varepsilon,t)\}.$$

Firstly, assume that $k\in\mathbb{N}-\{K_{\mu,1}(\varepsilon,t)\cup K_{\mu,2}(\varepsilon,t)\}$. From Definition 7-(e), we have

$$\mu(l_1-l_2,t)\geq\mu(x_k-l_1,\frac{t}{2})*\mu(x_k-l_2,\frac{t}{2})>(1-r)*(1-r)$$

and

$$\mu(l_1-l_2,t)>1-\varepsilon \tag{5}$$

holds. Since ε is arbitrary in (5), then $\mu(l_1-l_2)>1$ holds. Hence, from Definition 7-(c) it follows that $l_1=l_2$.

Secondly, assume that $k\in\mathbb{N}-\{K_{\nu,1}(\varepsilon,t)\cup K_{\nu,2}(\varepsilon,t)\}$. From the assumption and Definition 7-(k), we have

$$\nu(l_1-l_2,t)\leq\nu(x_k-l_1,\frac{t}{2})\diamond\mu(x_k-l_2,\frac{t}{2})<r\diamond r.$$

By using the fact $r\diamond r<\varepsilon$,

$$\nu(l_1-l_2,t)<\varepsilon \tag{6}$$

is obtained. Since ε is arbitrary in (6), then $l_1=l_2$ is obtained. Therefore, the limit of the sequence is unique. □

Theorem 4. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS and (x_n) be a sequence of X. If a sequence (x_n) is $D_p^qS(\mu,\nu)$-convergent, then it is $D_p^qS(\mu,\nu)$-Cauchy sequence in IFNS.*

Proof. Assume that the sequence (x_n) is $D_p^qS(\mu,\nu)$-convergent to $l\in X$. Let us choose $s>0$ so that $(1-\varepsilon)*(1-\varepsilon)>1-s$ and $\varepsilon\diamond\varepsilon<s$ hold for any $\varepsilon>0$. Then, for any $t>0$, we have

$$\delta_p^q\Big(\left\{k\in\mathbb{N}\,:\,\mu\left(x_k-l,t,\frac{t}{2}\right)\leq 1-\varepsilon \text{ or } \nu\left(x_k-l,\frac{t}{2}\right)\geq\varepsilon\right\}\Big)=0$$

and this implies that

$$\delta_p^q\Big(\left\{k\in\mathbb{N}\,:\,\mu\left(x_k-l,t,\frac{t}{2}\right)>1-\varepsilon \text{ or } \nu\left(x_k-l,\frac{t}{2}\right)<\varepsilon\right\}\Big)=0$$

Let $m\in\left\{k\in\mathbb{N}\,:\,\mu\left(x_k-l,t,\frac{t}{2}\right)>1-\varepsilon \text{ or } \nu\left(x_k-l,\frac{t}{2}\right)<\varepsilon\right\}$ be an arbitrary element. Let us denote $B(\varepsilon,t):=\left\{k\in\mathbb{N}\,:\,\mu\left(x_k-x_m,t,\frac{t}{2}\right)\leq 1-\varepsilon \text{ or } \nu\left(x_k-x_m,\frac{t}{2}\right)\geq\varepsilon\right\}$. It is enough to show that

$$B(\varepsilon,t)\subset\left\{k\in\mathbb{N}\,:\,\mu\left(x_k-l,t,\frac{t}{2}\right)\leq 1-\varepsilon \text{ or } \nu\left(x_k-l,\frac{t}{2}\right)\geq\varepsilon\right\}.$$

Let $k\in B(\varepsilon,t)-\left\{k\in\mathbb{N}\,:\,\mu\left(x_k-l,t,\frac{t}{2}\right)\leq 1-\varepsilon \text{ or } \nu\left(x_k-l,\frac{t}{2}\right)\geq\varepsilon\right\}$.

Then, we have $\mu(x_k - x_m, t) \leq 1-\varepsilon$ and $\mu(x_k - l, \frac{t}{2}) \leq 1-\varepsilon$, in particular $\mu(x_m - l, \frac{t}{2}) > 1-\varepsilon$. Hence,

$$1 - s \geq \mu(x_k - x_m, t) \geq \mu(x_k - l, \frac{t}{2}) * \mu(x_m - l, \frac{t}{2}) > (1-\varepsilon) * (1-\varepsilon) > 1 - s,$$

which is not possible. On the other hand, $\nu(x_k - x_m, t) \geq s$ and $\nu(x_k - l, \frac{t}{2}) < \varepsilon$, in particular $\nu(x_m - l, \frac{t}{2}) < \varepsilon$. Then,

$$s \geq \nu(x_k - x_m, t) \leq \nu(x_k - l, \frac{t}{2}) * \nu(x_m - l, \frac{t}{2}) < \varepsilon \diamond \varepsilon) < s,$$

which is not possible. This proves our claim. □

Theorem 5. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and (x_k) be a sequence of X. Then, $(x_n) \rightarrow l(D_p^q S(\mu, \nu))$ if and only if there exists a set $K = \{j_i \ : \ i \in \mathbb{N}\} \subset \mathbb{N}$ such that $\delta_p^q(K) = 1$ and $(x_n)_{n\in K} \rightarrow l(\mu, \nu)$.*

Proof. Necessity part: Assume that $(x_n) \rightarrow l(D_p^q S(\mu, \nu))$. Denote the following sets for any $t > 0$ and $s = 1, 2, \ldots$.

$$M_p^q(s,t) = \left\{ p(n) + 1 \leq k \leq q(n) \ : \ \mu(x_k - l, t) > 1 - \frac{1}{s} \text{ and } \nu(x_k - l, t) < \frac{1}{s} \right\}$$

and

$$K_p^q(s,t) = \left\{ p(n) + 1 \leq k \leq q(n) \ : \ \mu(x_k - l, t) \leq 1 - \frac{1}{s} \text{ and } \nu(x_k - l, t) \geq \frac{1}{s} \right\}.$$

Then, $\delta_p^q(K_p^q(s,t)) = 0$ holds because of $(x_n) \rightarrow l(D_p^q S(\mu, \nu)$. Hence, we have

$$M_p^q(s,t) \supset M_p^q(s+1,t)$$

and

$$\delta_p^q(M_p^q(s,t)) = 1 \tag{7}$$

for any $t > 0$ and $s = 1, 2, \ldots$. The last step of the proof of this part is to show for $n \in M_p^q(s,t)$

$$(x_n) \rightarrow l(\mu, \nu).$$

Suppose that this is not true, i.e., $(x_n) \nrightarrow l(\mu, \nu)$. From this assumption, there is $\alpha > 0$ and a positive natural number k_0 such that for all $k \geq k_0$,

$$\mu(x_k - l, t) \leq 1 - \alpha \ \textbf{ or } \ \nu(x_k - l, t) \geq \alpha$$

holds. It means that $\mu(x_k - l, t) > 1 - \alpha$ or $\nu(x_k - l, t) < \alpha$ holds for all $k < k_0$. Therefore, we have

$$\delta_p^q(\{k \in \mathbb{N} \ : \ \mu(x_k - l, t) > 1 - \alpha \text{ or } \nu(x_k - l, t) < \alpha\}) = 0.$$

Since $\alpha > \frac{1}{s}$, then $\delta_p^q(M_p^q(s,t)) = 0$. This is a contradiction of $x_n \rightarrow l(D_p^q S(\mu, \nu)$.

Sufficiency part: Let $K = \{k_j \ : \ j \in \mathbb{N}\} \subset \mathbb{N}$ such that $\delta_p^q(K) = 1$ and $x_{k_j} \to l(\mu,\nu)$. Hence, there exists $k_{j_0} \in \mathbb{N}$ such that for all $\alpha > 0$ and $t > 0$,

$$\mu(x_{k_j} - l, t) > 1 - \alpha \ \ \textbf{or} \ \ \nu(x_{k_j} - l, t) < \alpha$$

holds for all $k_j \geq k_{j_0}$. Also, a simple calculation gives that the following inclusion

$$\{k \in \mathbb{N} \ : \ \mu(x_k - l, t) \leq 1 - \alpha \ \text{ or } \ \nu(x_k - l, t) \geq \alpha\} \subset \mathbb{N} - \{k_{j_0}, k_{j_0} + 1, k_{j_0} + 2, \ldots\}$$

holds. So, we have

$$\delta_p^q(\{k \in \mathbb{N} \ : \ \mu(x_k - l, t) \leq 1 - \alpha \ \text{ or } \ \nu(x_k - l, t) \geq \alpha\}) \\ \leq \delta_p^q(\mathbb{N}) - \delta_p^q(\{k_{j_0}, k_{j_0} + 1, k_{j_0} + 2, \ldots\}).$$

This inequality completes the proof of the theorem. Hence, $(x_n) \to l(D_p^q S(\mu,\nu)$. □

By the proof of Theorem 5, the following result can be given for Cauchy sequences in IFNS:

Theorem 6. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS and (x_k) be a sequence of X. Then, (x_n) is a $D_p^q SC(\mu,\nu)$ Cauchy sequence if and only if there exists a set $K = \{j_i \ : \ i \in \mathbb{N}\} \subset \mathbb{N}$ such that $\delta_p^q(K) = 1$ and $(x_n)_{n \in K}$ is $D_p^q C(\mu,\nu)$.*

3 Comparison of $D_p^q(\mu,\nu)$ and $D_p^q S(\mu,\nu)$

In this section we deal with the relation between $D_p^q(\mu,\nu)$ and $D_p^q S(\mu,\nu)$ in an intuitionistic fuzzy normed space.

Theorem 7. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNS and (x_k) be a sequence of X. Then, $x_k \to l(D_p^q(\mu,\nu))$, implies $x_k \to l(D_p^q S(\mu,\nu))$.*

Proof. Assume $x_k \to l(D_p^q S(\mu,\nu))$. That is; for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} \tilde{\mu}(x_k - l, t) < \varepsilon \ \text{ and } \ \frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} \nu(x_k - l, t) < \varepsilon$$

are satisfied for all $n \geq n_0$. From the simple calculation we have following facts:

$$\frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} \tilde{\mu}(x_k - l, t) \geq \frac{|\{p(n) + 1 \leq k \leq q(n) \ : \ \tilde{\mu}(x_k - l, t) \geq \varepsilon\}|}{q(n) - p(n)} =: A_p^q(n)$$

$$\frac{1}{q(n) - p(n)} \sum_{p(n)+1}^{q(n)} \nu(x_k - l, t) \geq \frac{|\{p(n) + 1 \leq k \leq q(n) \ : \ \nu(x_k - l, t) \geq \varepsilon\}|}{q(n) - p(n)} =: B_p^q(n)$$

As a consequence of $x_k \to l(D_p^q(\mu,\nu))$ and the above inequalities, we have $\delta_p^q(A_p^q(n)) = 0$ and $\delta_p^q(B_p^q(n)) = 0$. Therefore,

$$\delta_p^q(\{n \ : \ \tilde{\mu}(x_k - l, t) \geq \varepsilon \ \text{ or } \ \nu(x_k - l, t) \geq \varepsilon\} = 0.$$

It means that $x_k \to l(D_p^q S(\mu,\nu))$. □

Remark 4. *The converse of Theorem 7 is not true, in general.*

For to see this, let us consider the space $(\mathbb{R}, \mu_0, \nu_0, *, \diamond)$ and the sequence $x = (x_k)$ as follows:

$$x_k := \begin{cases} k^2, & \left[\sqrt{q(n)}\right] - m_0 < k < \left[\sqrt{q(n)}\right], \quad n = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases},$$

where $q(n)$ is a monotone increasing sequence of positive integers and m_0 is a fixed positive natural number. If we consider D_p^q for the sequence $p(n)$ satisfying

$$0 < p(n) \leq \left[\sqrt{q(n)}\right] - m_0,$$

then we have

$$\frac{1}{q(n)-p(n)} |\{p(n)+1 \leq k \leq q(n), \;\; \mu_0(x_k, t) \geq 1-\varepsilon \;\text{ or }\; \nu_0(x_k, t) \geq \varepsilon\}| =$$

$$= \frac{1}{q(n)-p(n)} |\{p(n)+1 \leq k \leq q(n), \; |x_k| \geq \frac{t\varepsilon}{1-\varepsilon}\}| = \frac{m_0}{q(n)-p(n)},$$

which implies that $x_k \to l(D_p^q S(\mu, \nu))$. Also, the following inequality

$$\frac{1}{q(n)-p(n)} \sum_{p(n)+1}^{q(n)} \nu(x_k, t) = \frac{1}{q(n)-p(n)} \sum_{p(n)+1}^{q(n)} \frac{|x_k|}{t+|x_k|}$$

$$\geq \frac{1}{q(n)-p(n)} \sum_{p(n)+1}^{q(n)} \frac{(\left[\sqrt{q(n)}\right] - m_0)^2}{t + \left[\sqrt{q(n)}\right]^2} \geq \frac{(\left[\sqrt{q(n)}\right] - m_0)^2}{\left[\sqrt{q(n)}\right]^2}$$

holds. It means that the sequence (x_k) is not $D_p^q(\mu, \nu)$-convergent to zero because the left side of the above inequality tends to 1 when $n \to \infty$.

Let us recall that ℓ_∞ is the set of all bounded sequences. The following result shows that the converse of Theorem 7 is true for bounded sequences :

Theorem 8. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS and (x_k) be a sequence of X. If $x = (x_k) \in \ell_\infty$, then the convergence $x_k \to l(D_p^q S(\mu, \nu))$ implies that $x_k \to l(D_p^q(\mu, \nu))$.*

Proof. Suppose that $x = (x_k) \in \ell_\infty$ and $x_k \to l(DS[p, q])$. Under the assumption on (x_k) there exists a positive real number M, such that $\mu(x_k - l, t) > 1 - M$ and $\nu(x_k - l, t) < M$ hold for all k.

Therefore, the following inequality

$$\frac{1}{q(n)-p(n)} \sum_{p(n)+1}^{q(n)} \tilde{\mu}(x_k - l, t) = \frac{1}{q(n)-p(n)} \left\{ \left(\sum_{\substack{p(n)+1; \\ \tilde{\mu}(x_k - l, t) < \varepsilon}}^{q(n)} + \sum_{\substack{p(n)+1; \\ \tilde{\mu}(x_k - l, t) \geq \varepsilon}}^{q(n)} \right) \tilde{\mu}(x_k - l, t) \right\} <$$

$$< \varepsilon \frac{1}{q(n)-p(n)} |\{p(n)+1 \leq k \leq q(n) \;:\; \tilde{\mu}(x_k - l, t) < \varepsilon\}| +$$

$$+ M. \frac{1}{q(n)-p(n)} |\{p(n)+1 \leq k \leq q(n) \;:\; \tilde{\mu}(x_k - l, t) \geq \varepsilon\}|$$

holds. If we take limit by considering $x_k \to l(D_p^q S(\mu,\nu))$, then it is obtained that

$$\frac{1}{q(n)-p(n)} \sum_{p(n)+1}^{q(n)} \tilde{\mu}(x_k - l, t) > \varepsilon.$$

This gives

$$\frac{1}{q(n)-p(n)} \sum_{p(n)+1}^{q(n)} \tilde{\mu}(x_k - l, t) < 1-\varepsilon. \tag{8}$$

Also, the following inequality

$$\frac{1}{q(n)-p(n)} \sum_{p(n)+1}^{q(n)} \nu(x_k - l, t) = \frac{1}{q(n)-p(n)} \left\{ \left(\sum_{\substack{p(n)+1; \\ \tilde{\mu}(x_k-l,t)<\varepsilon}}^{q(n)} + \sum_{\substack{p(n)+1; \\ \tilde{\mu}(x_k-l,t)\geq\varepsilon}}^{q(n)} \right) \nu(x_k - l, t) \right\} <$$
$$< \varepsilon \frac{1}{q(n)-p(n)} |\{p(n)+1 \leq k \leq q(n) \;:\; \nu(x_k - l, t) < \varepsilon\}| +$$
$$+ M. \frac{1}{q(n)-p(n)} |\{p(n)+1 \leq k \leq q(n) \;:\; \nu(x_k - l, t) \geq \varepsilon\}|$$

holds. This gives that

$$\frac{1}{q(n)-p(n)} \sum_{p(n)+1}^{q(n)} \nu(x_k - l, t) < \varepsilon. \tag{9}$$

So, by considering (8) and (9), the proof of Theorem is completed. □

Theorem 9. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNS, (x_k) be a sequence of X and the sequence $\left(\frac{p(n)}{q(n)-p(n)}\right)$ is bounded. Then, $x_n \to l(S(\mu,\nu))$ implies $x_n \to l(D_p^q S(\mu,\nu))$.*

Proof. Since $\lim_{n\to\infty} q(n) = \infty$ and $x_n \to l(S(\mu,\nu))$, then

$$\lim_{n\to\infty} \frac{1}{q(n)} |\{k \leq q(n) \;:\; \mu(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon\}| = 0$$

holds from Theorem 2.2.1 in [13]. Also, the following inclusion

$$\{p(n)+1 \leq k \leq q(n) \;:\; \mu(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon\}$$
$$\subset \{k \leq q(n) \;:\; \mu(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon\}$$

and the inequality

$$|\{p(n)+1 \leq k \leq q(n) \;:\; \mu(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon\}|$$
$$\leq |\{k \leq q(n) \;:\; \mu(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon\}|$$

holds. So, we get

$$\frac{1}{q(n)-p(n)} |\{p(n)+1 \leq k \leq q(n) \;:\; \mu(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon\}|$$
$$\leq L(n). \frac{1}{q(n)} |\{k \leq q(n) \;:\; \mu(x_k - l, t) \leq 1-\varepsilon \text{ or } \nu(x_k - l, t) \geq \varepsilon\}|,$$

where $L(n) \left(\frac{q(n)}{q(n)-p(n)} \right)$. By taking limit when $n \to \infty$, we get $x_k \to l(D_p^q S(\mu,\nu))$. □

4 Comparison of $D_p^q S(\mu,\nu)$ and $D_r^h S(\mu,\nu)$

In this section we assume that

$$p(n) \leq r(n) < h(n) \leq q(n) \tag{10}$$

Theorem 10. *Let r and h be two non-negative sequences satisfying (10), such that $\{k,\ p(n) < k \leq r(n)\}$ and $\{k,\ h(n) < k \leq q(n)\}$ are finite sets for all $n \in \mathbb{N}$. Then, $x_k \to D_r^h S(\mu,\nu)$ implies that $x_k \to D_p^q S(\mu,\nu)$.*

Proof. It is clear that

$$\begin{aligned}\{p(n)+1 \leq k \leq q(n) \ : \ \mu(x_k-l,t) > 1-\varepsilon \text{ or } \nu(x_k-l,t) < \varepsilon\} = \\ = \{p(n) \leq k \leq r(n) \ : \ \mu(x_k-l,t) > 1-\varepsilon \text{ or } \nu(x_k-l,t) < \varepsilon\} \cup \\ \cup \{r(n) \leq k \leq h(n) \ : \ \mu(x_k-l,t) > 1-\varepsilon \text{ or } \nu(x_k-l,t) < \varepsilon\} \cup \\ \cup \{h(n) \leq k \leq q(n) \ : \ \mu(x_k-l,t) > 1-\varepsilon \text{ or } \nu(x_k-l,t) < \varepsilon\}\end{aligned}$$

is satisfied. Since the deferred density of a finite set is zero and $x_k \to D_r^h S(\mu,\nu)$, then we get the desired result. □

The converse of this theorem takes place under some conditions on p, q, h and r.

Theorem 11. *Assume that $\lim_{n\to\infty} \left(T(n) := \dfrac{q(n)-p(n)}{h(n)-r(n)} \right) = d > 0$. Then, $x_k \to l(D_p^q S(\mu,\nu))$ implies that $x_k \to l(D_r^h S(\mu,\nu))$.*

Proof. It is clear that the inclusion

$$\begin{aligned}\{r(n)+1 \leq k \leq h(n) \ : \ \mu(x_k-l,t) > 1-\varepsilon \text{ or } \nu(x_k-l,t) < \varepsilon\} \cup \\ \{p(n)+1 \leq k \leq q(n) \ : \ \mu(x_k-l,t) > 1-\varepsilon \text{ or } \nu(x_k-l,t) < \varepsilon\}\end{aligned}$$

is satisfied and we have

$$\begin{aligned}&\frac{1}{h(n)-r(n)} |\{r(n)+1 \leq k \leq h(n) \ : \ \mu(x_k-l,t) > 1-\varepsilon \text{ or } \nu(x_k-l,t) < \varepsilon\}| \\ &\quad \leq T(n) \frac{1}{q(n)-p(n)} |\{p(n)+1 \leq k \leq q(n) \ : \ \mu(x_k-l,t) > 1-\varepsilon \text{ or } \nu(x_k-l,t) < \varepsilon\}|.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \frac{1}{h(n)-r(n)} |\{r(n)+1 \leq k \leq h(n) \ : \ \mu(x_k-l,t) > 1-\varepsilon \text{ or } \nu(x_k-l,t) < \varepsilon\}| = 0,$$

which completes the proof. □

5 Conclusion

Since every usual norm defines an intuitionistic fuzzy norm, the results given here are more general than the results given in [13].

Also, some special cases of $p(n)$ and $q(n)$ in the method $D_p^q(\mu,\nu)$ and $D_p^q S(\mu,\nu)$ coincide with the corresponding results in [12, 16, 17].

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