ORIGINAL PAPER

SOME IDENTITIES OF GADOVAN NUMBERS

ORHAN DISKAYA[1], HAMZA MENKEN[2]

Manuscript received: 18.02.2020; Accepted paper: 03.04.2020;
Published online: 30.06.2020.

Abstract. *In this parer, we consider Padovan numbers with different initial values. We define the Gadovan numbers which generalizes a new class of Padovan numbers, and we derive Binet-like formulas, generating functions, exponential generating functions for the Gadovan numbers. Also, we obtain binomial sums, some identities and a matrix of the Gadovan numbers.*

Keywords: *Padovan numbers, Binet formula, generating function, binomial sum, matrix.*

1. INTRODUCTION

The Padovan sequence is named after Richard Padovan who attributed its discovery to Dutch architect Hans van der Laan in his 1994 essay Dom. Hans van der Laan : Modern Primitive. The sequence was described by Ian Stewart in his Scientific American column Mathematical Recreations in June 1996. He also writes about it in one of his books, "Math Hysteria: Fun Games With Mathematics".

In [4,5] the Padovan sequence $\{P_n\}$ is defined by a third order recurrence:

$$P_{n+3} = P_{n+1} + P_n, \quad n \geq 0 \tag{1}$$

with the different initial conditions $P_0 = 1$, $P_1 = 0$ and $P_2 = 1$. The first few members of this sequence is given as follow:

n	0	1	2	3	4	5	6	7	8	9	10	11	...
P_n	1	0	1	1	1	2	2	3	4	5	7	9	...

From [2], the recurrence (1) involves the characteristic equation

$$x^3 - x - 1 = 0 \tag{2}$$

If its roots are denoted by q_1, q_2 and q_3, then the following equalities can be derived:

$$q_1 + q_2 + q_3 = 0$$

$$q_1q_2 + q_1q_3 + q_2q_3 = -1$$

[1] Mersin University, Graduate School of Natural and Applied Sciences, Mersin, Turkey.
E-mail: orhandiskaya@mersin.edu.tr.
[2] Mersin University, Department of Mathematics , Mersin, Turkey. E-mail: hmenken@mersin.edu.tr.

$$q_1q_2q_3 = 1$$

Moreover, the Binet formula for the Padovan sequence is

$$P_n = p_1q_1^n + p_2q_2^n + p_3q_3^n \tag{3}$$

where

$$p_1 = \frac{q_2\,q_3 + 1}{(q_1 - q_2)(q_1 - q_3)}, \quad p_2 = \frac{q_1\,q_3 + 1}{(q_2 - q_1)(q_2 - q_3)} \quad \text{and} \quad p_3 = \frac{q_1q_2 + 1}{(q_3 - q_1)(q_3 - q_2)}.$$

The Padovan matrix is defined by [6]

$$Q_P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The n th power of Q_P gives

$$Q_P^n = \begin{bmatrix} P_{n-3} & P_{n-1} & P_{n-2} \\ P_{n-2} & P_n & P_{n-1} \\ P_{n-1} & P_{n+1} & P_n \end{bmatrix}.$$

Here, for convenience we take $P_{-1} = 0$ and $P_{-2} = 0$.

2. GADOVAN NUMBERS

The Gadovan sequence $\{GP_n\}$ is defined by the third order recurrence

$$GP_{n+3} = GP_{n+1} + GP_n, \quad n \geq 1 \tag{4}$$

with the initial conditions $GP_1 = a$, $GP_2 = b$ and $GP_3 = c$. The first few members of this sequence is given as follows

n	1	2	3	4	5	6	7	8	9	...
GP_n	a	b	c	$a+b$	$b+c$	$a+b+c$	$a+2b+c$	$a+2b+2c$	$2a+3b+2c$	...

Theorem 2.1. Let $\{GP_n\}$ be n th Gadovan number. Then,

$$GP_n = aP_{n-4} + bP_{n-2} + cP_{n-3}, \quad n \geq 4 \tag{5}$$

Proof: We establish this using principle of mathematical induction. Since,

$$GP_4 = aP_0 + bP_2 + cP_1 = a + b$$

and

$$GP_5 = aP_1 + bP_3 + cP_2 = b + c$$

the result is true for $n=4,5$. Assume that the relation is true for all positive integers $n \le k$. Then,

$$\begin{aligned} GP_{k+3} &= GP_{k+1} + GP_k \\ &= aP_{k-3} + bP_{k-1} + cP_{k-2} + aP_{k-4} + bP_{k-2} + cP_{k-3} \\ &= a\left(P_{k-3} + P_{k-4}\right) + b\left(P_{k-1} + P_{k-2}\right) + c\left(P_{k-2} + P_{k-3}\right) \\ &= aP_{k-1} + bP_{k+1} + cP_k \end{aligned}$$

Thus, by the strong version of principle of mathematical induction, the formula works for all positive integers n. Similar studies for Gibonacci which the generalization of Fibonacci numbers is available in [3] the book of Koshy Thomas. The proofs of the Theorem 2.2, Theorem 2.3, Theorem 2.5 and Theorem 2.6 can be given similar ways in [1,3,7].

Theorem 2.2. The Binet-like formula for the *n*th Gadovan number is

$$GP_n = p_1\hat{q}_1 q_1^n + p_2\hat{q}_2 q_2^n + p_3\hat{q}_3 q_3^n . \tag{6}$$

where

$$\hat{q}_1 = aq_1^{-4} + bq_1^{-2} + cq_1^{-3}, \quad \hat{q}_2 = aq_2^{-4} + bq_2^{-2} + cq_2^{-3} \text{ and } \hat{q}_3 = aq_3^{-4} + bq_3^{-2} + cq_3^{-3} .$$

Proof: Using Theorem 2.1, we have

$$\begin{aligned} GP_n &= aP_{n-4} + bP_{n-2} + cP_{n-3} \\ &= a\left(p_1 q_1^{n-4} + p_2 q_2^{n-4} + p_3 q_3^{n-4}\right) + b\left(p_1 q_1^{n-2} + p_2 q_2^{n-2} + p_3 q_3^{n-2}\right) + c\left(p_1 q_1^{n-3} + p_2 q_2^{n-3} + p_3 q_3^{n-3}\right) \\ &= p_1\left(aq_1^{-4} + bq_1^{-2} + cq_1^{-3}\right) q_1^n + p_2\left(aq_2^{-4} + bq_2^{-2} + cq_2^{-3}\right) q_2^n + p_3\left(aq_3^{-4} + bq_3^{-2} + cq_3^{-3}\right) q_3^n \\ &= p_1\hat{q}_1 q_1^n + p_2\hat{q}_2 q_2^n + p_3\hat{q}_3 q_3^n \end{aligned}$$

Theorem 2.3. The generating function of the Gadovan numbers is

$$G_{GP}(x) = \sum_{n=1}^{\infty} GP_n x^n = \frac{ax + bx^2 + (c-a)x^3}{1 - x^2 - x^3} .$$

Proof: Assume that the function

$$G_{GP}(x) = \sum_{n=1}^{\infty} GP_n x^n = GP_1 x + GP_2 x^2 + \cdots + GP_n x^n + \cdots$$

be the generating function of the Gadovan numbers. Multiply both of side of the equality by the term $-x^2$ such as

$$-x^2 G_{GP}(x) = -GP_1 x^3 - GP_2 x^4 - \cdots - GP_n x^{n+2} - \cdots$$

and that is multiplied every side with $-x^3$ such as

$$-x^3 G_{GP}(x) = -GP_1 x^4 - GP_2 x^5 - \cdots - GP_n x^{n+3} - \cdots .$$

Then, we write

$$\left(1-x^2-x^3\right)G_{GP}\left(x\right)=GP_1x+GP_2x^2+\left(GP_3-GP_1\right)x^3+\left(GP_4-GP_2-GP_1\right)x^4+\cdots$$
$$+\left(GP_n-GP_{n-2}-GP_{n-3}\right)x^n+\cdots$$

Now, by using

$$GP_1=a,\ GP_2=b,\ GP_3=c,\ GP_4=a+b,\ GP_5=b+c,\ ...$$

We obtain that

$$G_{GP}\left(x\right)=\frac{ax+bx^2+\left(c-a\right)x^3}{1-x^2-x^3}.$$

Thus, the proof is completed.

Theorem 2.4. The exponential generating function for the nth Gadovan numbers is

$$E_{GP}\left(x\right)=\sum_{n=1}^{\infty}\frac{GP_n}{n!}x^n=p_1\hat{q}_1e^{q_1x}+p_2\hat{q}_2e^{q_2x}+p_3\hat{q}_3e^{q_3x}.$$

Proof: We know that,

$$e^{q_1x}=\sum_{n=1}^{\infty}\frac{q_1^n}{n!}x^n,\ e^{q_2x}=\sum_{n=1}^{\infty}\frac{q_2^n}{n!}x^n \text{ and } e^{q_3x}=\sum_{n=1}^{\infty}\frac{q_3^n}{n!}x^n$$

Let's multiply each side of the first equation by $p_1\hat{q}_1$, the second equation by $p_2\hat{q}_2$ and the third equation by $p_3\hat{q}_3$. Then, we added all equations. So, the following equation is obtained.

$$p_1\hat{q}_1e^{q_1x}+p_2\hat{q}_2e^{q_2x}+p_3\hat{q}_3e^{q_3x}=\sum_{n=1}^{\infty}\frac{p_1\hat{q}_1q_1^n+p_2\hat{q}_2q_2^n+p_3\hat{q}_3q_3^n}{n!}x^n=\sum_{n=1}^{\infty}\frac{GP_n}{n!}x^n.$$

Thus, the proof is completed.

Theorem 2.5. Let $m,n\in\mathbb{Z}^+$. Then,

$$\sum_{n=1}^{m}\binom{m}{n}GP_n=GP_{3m}$$

Proof: Applying Binet-like formula (6) and combining this with (2) we obtain the identity

$$\begin{aligned}
\sum_{n=1}^{m}\binom{m}{n}GP_n &= \sum_{n=1}^{m}\binom{m}{n}\left(p_1\hat{q}_1q_1^n + p_2\hat{q}_2q_2^n + p_3\hat{q}_3q_3^n\right) \\
&= p_1\hat{q}_1\left(\sum_{n=1}^{m}\binom{m}{n}q_1^n1^{m-n}\right) + p_2\hat{q}_2\left(\sum_{n=1}^{m}\binom{m}{n}q_2^n1^{m-n}\right) + p_3\hat{q}_3\left(\sum_{n=1}^{m}\binom{m}{n}q_3^n1^{m-n}\right) \\
&= p_1\hat{q}_1\left(q_1+1\right)^m + p_2\hat{q}_2\left(q_2+1\right)^m + p_3\hat{q}_3\left(q_3+1\right)^m \\
&= p_1\hat{q}_1q_1^{3m} + p_2\hat{q}_2q_2^{3m} + p_3\hat{q}_3q_3^{3m}.
\end{aligned}$$

Thus, the proof is completed.

Theorem 2.6. Let $m,n \in \mathbb{Z}^+$. Then,

$$\sum_{k=1}^{m}\binom{m}{k}GP_{n-k} = GP_{n+2m}$$

Proof: Applying Binet-like formula (6) and combining this with (2) we obtain the identity

$$\begin{aligned}
\sum_{k=1}^{m}\binom{m}{k}GP_{n-k} &= \sum_{k=1}^{m}\binom{m}{k}\left(p_1\hat{q}_1q_1^{n-k} + p_2\hat{q}_2q_2^{n-k} + p_3\hat{q}_3q_3^{n-k}\right) \\
&= p_1\hat{q}_1\left(\sum_{k=1}^{m}\binom{m}{k}1^k q_1^{m-k}\right)q_1^{n-m} + p_2\hat{q}_2\left(\sum_{k=1}^{m}\binom{m}{k}1^k q_2^{m-k}\right)q_2^{n-m} \\
&+ p_3\hat{q}_3\left(\sum_{k=1}^{m}\binom{m}{k}1^k q_3^{m-k}\right)q_3^{n-m} \\
&= p_1\hat{q}_1\left(q_1+1\right)^m q_1^{n-m} + p_2\hat{q}_2\left(q_2+1\right)^m q_2^{n-m} + p_3\hat{q}_3\left(q_3+1\right)^m q_3^{n-m} \\
&= p_1\hat{q}_1q_1^{n+2m} + p_2\hat{q}_2q_2^{n+2m} + p_3\hat{q}_3q_3^{n+2m}.
\end{aligned}$$

Thus, the proof is completed.

Theorem 2.7. For all $n \geq 1$,

$$\begin{bmatrix} GP_{n-3} & GP_{n-1} & GP_{n-2} \\ GP_{n-2} & GP_n & GP_{n-1} \\ GP_{n-1} & GP_{n+1} & GP_n \end{bmatrix} = \begin{bmatrix} P_{n-3} & P_{n-1} & P_{n-2} \\ P_{n-2} & P_n & P_{n-1} \\ P_{n-1} & P_{n+1} & P_n \end{bmatrix}\begin{bmatrix} c-a & b & a \\ a & c & b \\ b & a+b & c \end{bmatrix}.$$

Proof: Applying (5) and combining this with (1) and (4) we obtain the identity

$$\begin{bmatrix} A & B & C \\ X & Y & Z \\ K & L & M \end{bmatrix} = \begin{bmatrix} P_{n-3} & P_{n-1} & P_{n-2} \\ P_{n-2} & P_n & P_{n-1} \\ P_{n-1} & P_{n+1} & P_n \end{bmatrix}\begin{bmatrix} c-a & b & a \\ a & c & b \\ b & a+b & c \end{bmatrix}$$

Then, we write

$$A = (c-a)P_{n-3} + aP_{n-1} + bP_{n-2} = cP_{n-3} - aP_{n-3} + a\left(P_{n-3} + P_{n-4}\right) + bP_{n-2} = aP_{n-4} + bP_{n-2} + cP_{n-3} = GP_{n-3}$$

$$B = bP_{n-3} + cP_{n-1} + \left(a+b\right)P_{n-2} = bP_{n-3} + cP_{n-1} + aP_{n-2} + bP_{n-2} = aP_{n-2} + bP_{n} + cP_{n-1} = GP_{n-1}$$

$$C = aP_{n-3} + bP_{n-1} + cP_{n-2} = GP_{n-2}$$

$$X = \left(c-a\right)P_{n-2} + aP_{n} + bP_{n-1} = cP_{n-2} - aP_{n-2} + a\left(P_{n-2} + P_{n-3}\right) + bP_{n-1} = aP_{n-3} + bP_{n-1} + cP_{n-2} = GP_{n-2}$$

$$Y = bP_{n-1} + cP_{n} + \left(a+b\right)P_{n-1} = bP_{n-2} + cP_{n} + aP_{n-1} + bP_{n-1} = aP_{n-1} + bP_{n+1} + cP_{n} = GP_{n}$$

$$Z = aP_{n-2} + bP_{n} + cP_{n-1} = GP_{n-1}$$

$$K = \left(c-a\right)P_{n-1} + aP_{n+1} + bP_{n} = cP_{n-1} - aP_{n-1} + a\left(P_{n-1} + P_{n-2}\right) + bP_{n} = aP_{n-2} + bP_{n} + cP_{n-1} = GP_{n-1}$$

$$L = bP_{n-1} + cP_{n+1} + \left(a+b\right)P_{n} = bP_{n-1} + cP_{n+1} + aP_{n} + bP_{n} = aP_{n} + bP_{n+2} + cP_{n+1} = GP_{n+1}$$

$$M = aP_{n-1} + bP_{n+1} + cP_{n} = GP_{n}$$

So,

$$\begin{bmatrix} A & B & C \\ X & Y & Z \\ K & L & M \end{bmatrix} = \begin{bmatrix} GP_{n-3} & GP_{n-1} & GP_{n-2} \\ GP_{n-2} & GP_{n} & GP_{n-1} \\ GP_{n-1} & GP_{n+1} & GP_{n} \end{bmatrix}.$$

3. CONCLUSION

In this paper, we define and study the Gadovan numbers as a generalization of the Padovan numbers. We obtain some results including recurrence relations, summation formulas, binomial sums and Binet's formulas. We derive their generating and exponential generating functions. We establish a matrix in term of the Gadovan numbers.

REFERENCES

[1] Ipek, A., *Advances in Applied Cliford Algebras*, **27**(2), 1343, 2017.
[2] Jiang, Z., Shen, N., Li, J., *Journal of Applied Mathematics*, **2014**, 585438, 2014.
[3] Koshy, T., *Fibonacci and Lucas Numbers with Applications*, Vol. 1, John Wiley Sons, New Jersey, 2018.
[4] Tasci, D., *Journal of Science and Arts*, **1**(42), 125, 2018.
[5] Yilmaz, N., Taskara, N., *Abstract and Applied Analysis*, **2013**, 497418, 2013.
[6] Sokhuma, K., Applied *Mathematical Sciences*, **7**(142), 7093, 2013.
[7] Dişkaya, O., Menken, H., *Journal of Advanced Mathematical* Studies, **12**(2), 186, 2019.