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ON THE NOVEL GENERALIZATIONS OF THE PADOVAN SEQUENCE

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Abstract. In the present work, we consider the Padovan sequence and define a sequence(called Quadrovan) that is a new generalization. In addition, we give the previously defined Tridovan sequence as a generalization of the Padovan sequence. We derive the Binet-like formulas, the generating functions and the exponential generating functions for the Tridovan and Quadrovan sequences. Also, we establish their series and matrices.

1. INTRODUCTION

Integer sequences are one of the subjects with the most applications area in mathematics. The Fibonacci sequence is the most famous one of integer sequences. The Fibonacci sequence has applications in many branches of science such as nature, anatomy, botany, zoology, art, music, analysis, physics, astronomy, chemistry, biology and computers. The positive real root of the characteristic equation of Fibonacci numbers gives the golden ratio which has many applications in science, art and architecture. Many scientists deal with the Fibonacci sequence and its generalizations in recent years. Some of these generalizations are number sequences such as Lucas, Pell and Jacobsthal [5–7]. Fibonacci sequence and these generalizations have the same second-order characteristic equation. The Padovan sequence and similar generalizations have a third-order characteristic equation. We begin the study by giving the definition of the Padovan sequence.
The Padovan sequence $\{P_n\}$ is defined by the third order recurrence

$$P_{n+3} = P_{n+1} + P_n, \quad n \geq 0 \tag{1.1}$$

with the initial conditions $P_0 = 1$, $P_1 = 0$ and $P_2 = 1$. The first few values of this sequence are 1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265.
The recurrence (1.1) involves the characteristic equation

$$x^3 - x - 1 = 0 \tag{1.2}$$

The roots of the equation (1.2) are

$$a_1 \approx 1.3247,$$
$$a_2 \approx -0.66236 - 0.56228i,$$
$$a_3 \approx -0.66236 + 0.56228i,$$

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Moreover, the Binet-like formula for the Padovan sequence is

$$P_n = aa_1^n + ba_2^n + ca_3^n \tag{1.3}$$

where,

$$a = \frac{a_2a_3+1}{(a_1-a_2)(a_1-a_3)}, \quad b = \frac{a_1a_3+1}{(a_2-a_1)(a_2-a_3)} \quad and \quad c = \frac{a_1a_2+1}{(a_3-a_1)(a_3-a_2)}.$$

Sokhuma [13] have defined Padovan $Q-$matrix as

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

and established that

$$Q^n = \begin{pmatrix} P_{n-3} & P_{n-1} & P_{n-2} \\ P_{n-2} & P_n & P_{n-1} \\ P_{n-1} & P_{n+1} & P_n \end{pmatrix} \quad n \geq 0.$$

Yilmaz and Taskara [12] have defined some matrix sequences in terms of Padovan and Perrin numbers. Feinberg [3] have defined Tribonacci numbers and given some of their properties. Hoggatt and Bicknell [4] have investigated the Tribonacci polynomials.
The Tridovan sequence $\{T_n\}$ is defined in [11] by the fourth-order recurrence

$$TP_{n+4} = TP_{n+2} + TP_{n+1} + TP_n, \quad n \geq 0 \tag{1.4}$$

with the initial conditions $TP_0 = 1$, $TP_1 = 0$, $TP_2 = 1$ and $TP_3 = 1$. The first few values of this sequence are 1, 0, 1, 1, 2, 2, 4, 5, 8, 11, 17, 24, 36, 52, 87.
The recurrence (1.4) involve the characteristic equation

$$y^4 - y^2 - y - 1 = 0. \tag{1.5}$$

The graph of the equation $g(y) = y^4 - y^2 - y - 1$ is as follows

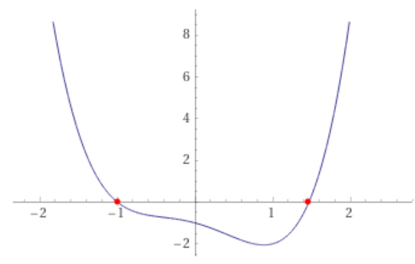

FIGURE 1

The roots in the complex plane of the equation (1.5) is as follows

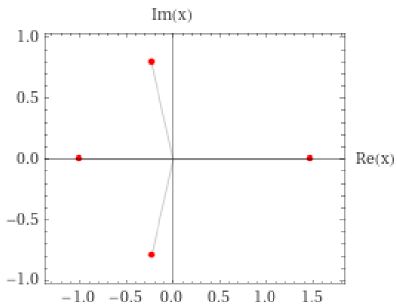

FIGURE 2

Means that the roots of the equation (1.5) are

$$\begin{aligned} b_1 &\approx -1, \\ b_2 &\approx 1.4656, \\ b_3 &\approx -0.23279 - 0.79255i, \\ b_4 &\approx -0.23279 + 0.79255i. \end{aligned}$$

2. The Tridovan and Quadrovan Sequences

A few studies on $k-$generalization of the Fibonacci sequence (or $k-$bonacci) can be found in [1,2,8–10]. Inspired by these studies, we consider new generalizations of the Padovan numbers and give similar investigations.

Definition 2.1. *The Quadrovan sequence $\{QP_n\}$ is defined by fifth-order recurrence*

$$QP_{n+5} = QP_{n+3} + QP_{n+2} + QP_{n+1} + QP_n, \tag{2.6}$$

with the initial conditions $QP_0 = 1$, $QP_1 = 0$, $QP_2 = 1$, $QP_3 = 1$ and $QP_4 = 2$.

The first few members of Quadrovan sequence are 1, 0, 1, 1, 2, 3, 4, 7, 10, 16, 24, 37. The recurrence (2.6) involve the characteristic equation

$$z^5 - z^3 - z^2 - z - 1 = 0. \tag{2.7}$$

The graph of the function $f(z) = z^5 - z^3 - z^2 - z - 1$ is as follows

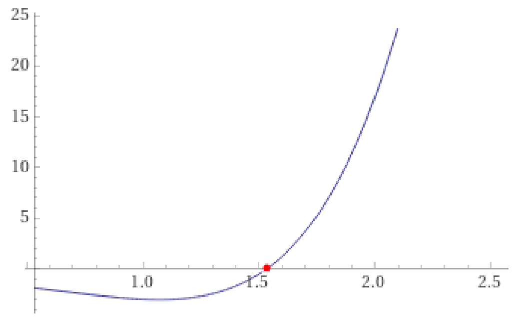

FIGURE 3

The roots in the complex plane of the equation (2.7) is as follows

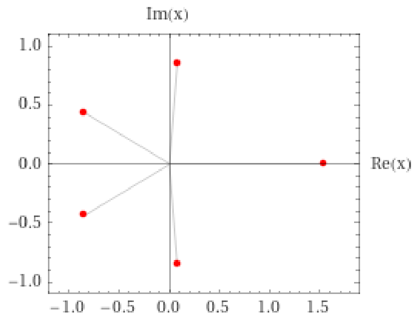

FIGURE 4

Means that the roots of the equation (2.7) are

$$\begin{aligned} c_1 &= 1.53416, \\ c_2 &= -0.847177 - 0.431759i, \\ c_3 &= -0.847177 + 0.431759i, \\ c_4 &= 0.0800981 - 0.845298i, \\ c_5 &= 0.0800981 + 0.845298i, \end{aligned}$$

Theorem 2.1. *The Binet formulas for the Tridovan and Quadrovan sequences are, respectively,*

1.

$$TP_n = t_1 b_1^n + t_2 b_2^n + t_3 b_3^n + t_4 b_4^n,$$

where

$$t_1 = \frac{-b_1 - b_3 - b_2 b_3 b_4 - b_4 + 1}{b_1^3 - b_1^2 b_2 - b_1^2 b_3 + b_1 b_2 b_3 - b_1^2 b_4 + b_1 b_2 b_4 - b_1 b_3 b_4 + b_2 b_3 b_4}$$

$$t_2 = \frac{-b_1 - b_3 - b_1b_3b_4 - b_4 + 1}{b_2^3 - b_1b_2^2 - b_2^2b_3 + b_1b_2b_3 - b_2^2b_4 + b_1b_2b_4 - b_1b_3b_4 + b_2b_3b_4}$$
$$t_3 = \frac{-b_1 - b_2 - b_1b_2b_4 - b_4 + 1}{b_3^3 - b_1b_3^2 - b_2b_3^2 + b_1b_2b_3 - b_1b_2b_4 - b_3^2b_4 + b_1b_3b_4 + b_2b_3b_4}$$
$$t_2 = \frac{-b_1 - b_2 - b_1b_2b_3 - b_3 + 1}{b_4^3 - b_1b_4^2 - b_2b_4^2 + b_3b_4^2 + b_1b_2b_4 + b_1b_3b_4 - b_1b_2b_3 + b_2b_3b_4}.$$

2.
$$QP_n = q_1c_1^n + q_2c_2^n + q_3c_3^n + q_4c_4^n + q_5c_5^n,$$
where q_1 q_2, q_3, q_4 *and* q_5 *are constants that only depend on* c_1, c_2, c_3, c_4 *and* c_5.

Proof. The proof is easily performed with the Cramer Rule or the Gauss–Jordan Elimination Method. □

Theorem 2.2. *The generating functions for the Tridovan and Quadrovan sequences are, respectively,*

1.
$$G_{TP}(x) = \sum_{n=0}^{\infty} TP_n x^n = \frac{1}{1 - x^2 - x^3 - x^4},$$

2.
$$G_{QP}(x) = \sum_{n=0}^{\infty} QP_n x^n = \frac{1}{1 - x^2 - x^3 - x^4 - x^5},$$

Proof. We prove only the second assertion. The first assertion can be proved similarly. Let
$$G_{QP}(x) = \sum_{n=0}^{\infty} QP_n x^n = QP_0 + QP_1x + QP_2x^2 + \cdots + QP_nx^n + \dots$$
be the generating function of the Quadrovan sequence. Multiply both of side of the equality by the term $-x^2$, $-x^3$, $-x^4$, $-x^5$, respectively, such as
$$-x^2G_{QP}(x) = -QP_0x^2 - QP_1x^3 - QP_2x^4 - \cdots - QP_nx^{n+2} - \dots$$
$$-x^3G_{QP}(x) = -QP_0x^3 - QP_1x^4 - QP_2x^5 - \cdots - QP_nx^{n+3} - \dots$$
$$-x^4G_{QP}(x) = -QP_0x^4 - QP_1x^5 - QP_2x^6 - \cdots - QP_nx^{n+4} - \dots$$
$$-x^5G_{QP}(x) = -QP_0x^5 - QP_1x^6 - QP_2x^7 - \cdots - QP_nx^{n+5} - \dots$$
Then, we write
$$\begin{aligned}(1 - x^2 - x^3 - x^4 - x^5)G_{QP}(x) &= QP_0 + QP_1x + (QP_2 - QP_0)x^2 \\ &+ (QP_3 - QP_1 - QP_0)x^3 \\ &+ (QP_4 - QP_2 - QP_1 - QP_0)x^4 \\ &+ (QP_5 - QP_3 - QP_2 - QP_1 - QP_0)x^5 + \dots \\ &+ (QP_n - QP_{n-2} - QP_{n-3} - QP_{n-4} - QP_{n-5})x^n + \dots\end{aligned}$$

Now, by using the initial conditions of the Quadrovan sequence and

$$QP_n - QP_{n-2} - QP_{n-3} - QP_{n-4} - QP_{n-5} = 0,$$

we obtain that

$$G_{QP}(x) = \frac{1}{1 - x^2 - x^3 - x^4 - x^5}.$$

Thus, the proof is completed. □

Theorem 2.3. *The exponential generating functions for the Tridovan and Quadrovan sequences are, respectively,*

1.
$$E_{TP}(x) = \sum_{n=0}^{\infty} \frac{TP_n}{n!} x^n = t_1 e^{b_1 x} + t_2 e^{b_2 x} + t_3 e^{b_3 x} + t_4 e^{b_4 x},$$

2.
$$E_{QP}(x) = \sum_{n=0}^{\infty} \frac{QP_n}{n!} x^n = q_1 e^{c_1 x} + q_2 e^{c_2 x} + q_3 e^{c_3 x} + q_4 e^{c_4 x} + q_5 e^{c_5 x},$$

Proof. We prove only the second assertion. The first assertion can be proved similarly.
We know that,
$e^{c_1 x} = \sum_{n=0}^{\infty} \frac{c_1^n x^n}{n!}$, $e^{c_2 x} = \sum_{n=0}^{\infty} \frac{c_2^n x^n}{n!}$, $e^{c_3 x} = \sum_{n=0}^{\infty} \frac{c_3^n x^n}{n!}$, $e^{c_4 x} = \sum_{n=0}^{\infty} \frac{c_4^n x^n}{n!}$, $e^{c_5 x} = \sum_{n=0}^{\infty} \frac{c_5^n x^n}{n!}$.
Multiplying each side of the identities, respectively, by q_1, q_2, q_3, q_4, q_5 and adding of them, we obtain that

$$\begin{aligned} E_{QP}(x) &= q_1 e^{c_1 x} + q_2 e^{c_2 x} + q_3 e^{c_3 x} + q_4 e^{c_4 x} + q_5 e^{c_5 x} \\ &= \sum_{n=0}^{\infty} (q_1 c_1^n + q_2 c_2^n + q_3 c_3^n + q_4 c_4^n + q_5 c_5^n) \frac{1}{n!} x^n \\ &= \sum_{n=0}^{\infty} \frac{QP_n}{n!} x^n. \end{aligned}$$

□

Theorem 2.4. *The series for the Tridovan and Quadrovan sequences are, respectively,*

1.
$$S_{TP}(x) = \sum_{n=0}^{\infty} \frac{TP_n}{x^n} = \frac{x^4}{x^4 - x^2 - x - 1},$$

2.
$$S_{QP}(x) = \sum_{n=0}^{\infty} \frac{QP_n}{x^n} = \frac{x^5}{x^5 - x^3 - x^2 - x - 1},$$

Proof. We prove only the second assertion. The first assertion can be proved similarly.
Let

$$S_{QP}(x) = \sum_{n=0}^{\infty} \frac{QP_n}{x^n} = QP_0 + \frac{QP_1}{x} + \frac{QP_2}{x^2} + \frac{QP_3}{x^3} + \cdots + \frac{QP_n}{x^n} + \ldots$$

be the series of the Quadrovan sequence. Multiply both of side of the equality by the term x^5, $-x^3$ $-x^2$, $-x$, -1, respectively, such as

$$\begin{aligned}
x^5 S_{QP}(x) &= QP_0 x^5 + QP_1 x^4 + QP_2 x^3 + \cdots + QP_n x^{5-n} + \dots \\
x^3 S_{QP}(x) &= QP_0 x^3 + QP_1 x^2 + QP_2 x + \cdots + QP_n x^{3-n} + \dots \\
x^2 S_{QP}(x) &= QP_0 x^2 + QP_1 x^1 + QP_2 + \cdots + QP_n x^{2-n} + \dots \\
x S_{QP}(x) &= QP_0 x + QP_1 + QP_2 x^{-1} + \cdots + QP_n x^{1-n} + \dots
\end{aligned}$$

Then, we write

$$\begin{aligned}
(x^5 - x^3 - x^2 - x - 1) S_{QP}(x) = {} & QP_0 x^5 + QP_1 x^4 + (QP_2 - QP_0) x^3 \\
& + (QP_3 - QP_1 - QP_0) x^2 + (QP_4 - QP_2 - QP_1 - QP_0) x + \dots \\
& + (QP_n - QP_{n-2} - QP_{n-3} - QP_{n-4} - QP_{n-5}) x^{5-n} + \dots
\end{aligned}$$

Now, by using the initial conditions of the Quadrovan sequence and

$$QP_n - QP_{n-2} - QP_{n-3} - QP_{n-4} - QP_{n-5} = 0,$$

we obtain that

$$S_{QP}(x) = \frac{x^5}{x^5 - x^3 - x^2 - x - 1}.$$

Thus, the proof is completed. □

In this part of the study, we investigate the new property of the Quadrovan numbers in relation to the Quadrovan $Mq-$matrix

$$Mq = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

More generally, we have

$$Mq^n = \begin{pmatrix} QP_{n-5} & QP_{n-5} + QP_{n-6} & QP_{n-5} + QP_{n-6} + QP_{n-7} & QP_{n-3} & QP_{n-4} \\ QP_{n-4} & QP_{n-4} + QP_{n-5} & QP_{n-4} + QP_{n-5} + QP_{n-6} & QP_{n-2} & QP_{n-3} \\ QP_{n-3} & QP_{n-3} + QP_{n-4} & QP_{n-3} + QP_{n-4} + QP_{n-5} & QP_{n-1} & QP_{n-2} \\ QP_{n-2} & QP_{n-2} + QP_{n-3} & QP_{n-2} + QP_{n-3} + QP_{n-4} & QP_n & QP_{n-1} \\ QP_{n-1} & QP_{n-1} + QP_{n-2} & QP_{n-1} + QP_{n-2} + QP_{n-3} & QP_{n+1} & QP_n \end{pmatrix}$$

This strategy allows us to obtain new relations for the Quadrovan sequences.We can express the Tridovan sequences in a similar way.

Theorem 2.5. *The matrices for the Tridovan and Quadrovan sequences are, respectively,*

1.

$$Mt^n = \begin{pmatrix} TP_{n-4} & TP_{n-4} + TP_{n-5} & TP_{n-2} & TP_{n-3} \\ TP_{n-3} & TP_{n-3} + TP_{n-4} & TP_{n-1} & TP_{n-2} \\ TP_{n-2} & TP_{n-2} + TP_{n-3} & TP_n & TP_{n-1} \\ TP_{n-1} & TP_{n-1} + TP_{n-2} & TP_{n+1} & TP_n \end{pmatrix},$$

2

$$Mq^n = \begin{pmatrix} QP_{n-5} & QP_{n-5}+QP_{n-6} & QP_{n-5}+QP_{n-6}+QP_{n-7} & QP_{n-3} & QP_{n-4} \\ QP_{n-4} & QP_{n-4}+QP_{n-5} & QP_{n-4}+QP_{n-5}+QP_{n-6} & QP_{n-2} & QP_{n-3} \\ QP_{n-3} & QP_{n-3}+QP_{n-4} & QP_{n-3}+QP_{n-4}+QP_{n-5} & QP_{n-1} & QP_{n-2} \\ QP_{n-2} & QP_{n-2}+QP_{n-3} & QP_{n-2}+QP_{n-3}+QP_{n-4} & QP_{n} & QP_{n-1} \\ QP_{n-1} & QP_{n-1}+QP_{n-2} & QP_{n-1}+QP_{n-2}+QP_{n-3} & QP_{n+1} & QP_{n} \end{pmatrix}$$

Proof. We prove only the second assertion. The first assertion can be proved similarly. We establish this using principle of mathematical induction. Since,

$$Mq^1Mq^1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix} = Mq^2.$$

The result is true for $n = 1$. Assume that the relation is true for all positive integers $n \leq k$. Then

$$Mq^kMq^1 = Mq^k \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} = Mq^{k+1}$$

Thus, by the strong version of principle of mathematical induction, the formula works for all positive integers n . □

Theorem 2.6. *The inverse of matrices for the Tridovan and Quadrovan sequences are, respectively,*

1.

$$(Mt^n)^{-1} = Mt^{-n}$$

2

$$(Mq^n)^{-1} = Mq^{-n}$$

Proof. We prove only the second assertion. The first assertion can be proved similarly. We establish this using principle of mathematical induction. Since,

$$Mq^1(Mq^1)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = Mr^0 = I.$$

The result is true for $n = 1$. Assume that the relation is true for all positive integers $n \leq k$. Then

$$Mq^{k+1}(Mq^{k+1})^{-1} = Mq^kMr^1(Mq^1)^{-1}(Mq^k)^{-1} = I$$

Thus, by the strong version of principle of mathematical induction, the formula works for all positive integers n. □

3. Conclusions

The present work deals with the Padovan sequence and some new generalizations. It is well known that the Padavon sequence produces the plastic ratio which is used in art, architecture and applied science. In this work, we define some new generalizations of the Padovan sequence. We derive the Binet-like formulas, the generating functions and the exponential generating functions for the new generalizations of the Padovan sequence. Finally, we establish a relationship between their matrices and inverses.

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