International Journal of Geometric Methods in Modern Physics
Vol. 14, No. 12 (2017) 1750171 (16 pages)

DOI: 10.1142/S0219887817501717

Scalar curvature of Lagrangian Riemannian submersions and their harmonicity

Şemsi Eken Meriç
Department of Mathematics, Faculty of Science, Mersin University, 33343, Mersin, Turkey
semsieken@hotmail.com

Erol Kiliç
Department of Mathematics, Faculty of Science, İnönü University, 44280, Malatya, Turkey
erol.kilic@inonu.edu.tr

Yasemin Sağiroğlu
Department of Mathematics, Faculty of Science, Karadeniz Technical University, 61080, Trabzon, Turkey
sagiroglu.yasemin@gmail.com

Received 27 April 2016
Accepted 31 July 2017
Published 29 August 2017

We dedicate our paper to Professor Sadık Keleş on the occasion of his 65th birthday.

In this paper, we consider a Lagrangian Riemannian submersion from a Hermitian manifold to a Riemannian manifold and establish some basic inequalities to obtain relationships between the intrinsic and extrinsic invariants for such a submersion. Indeed, using these inequalities, we provide necessary and sufficient conditions for which a Lagrangian Riemannian submersion π has totally geodesic or totally umbilical fibers. Moreover, we study the harmonicity of Lagrangian Riemannian submersions and obtain a characterization for such submersions to be harmonic.

Keywords: Curvature; Hermitian manifold; Riemannian submersion.

Mathematics Subject Classification 2010: 53C21, 53C55, 53C40

1. Introduction

The study of Riemannian submersions was initiated by O'Neill and Gray, independently, and formulated the basis theory of Riemannian submersions. They defined two tensor fields which are known as fundamental tensor fields or O'Neill tensors in the literature. O'Neill obtained a set of five equations which are analogues of Gauss, Codazzi and Ricci equations for submanifolds. This theory has been extended in the last three decades. For instance, Watson first introduced almost Hermitian submersions between almost Hermitian manifolds in [27]. Then, such submersions have been developed to Kähler submersions, Lagrangian Riemannian submersions, CR-submersions, etc. (for details, see [1, 15–17]). Nowadays, this theory is still an active field in differential geometry. In [19], Şahin introduced anti-invariant Riemannian submersions and obtained some characterizations for Lagrangian Riemannian submersions which are special anti-invariant Riemannian submersions. Then, he defined different kinds of submersions such as slant submersions, semi-invariant Riemannian submersions, etc. (see [20–22]).

In [23], Taştan studied Lagrangian submersion and showed that the horizontal distribution of a Lagrangian Riemannian submersion from a Kähler manifold to a Riemannian manifold is integrable. Later, Taştan, Şahin, and Yanan introduced the notion of hemi-slant submersions to generalize anti-invariant submersions (for details, see [24]).

The theory of Riemannian submersions is a very interesting topic in differential geometry, since it has many applications in physics and mechanics. For instance, in the theory of Kaluza–Klein, one starts with the hypothesis that the space-time has $(4+m)$-dimensions. An interesting mechanism for space-time compactification is proposed in the form of a nonlinear sigma model. The general solutions of this model can be expressed in terms of harmonic maps satisfying the Einstein equations. A very general class of solutions is given by Riemannian submersions of this model (see [15, Chap. 9]). Moreover, the notion of Riemannian submersion have played an important role for the modeling and controlling of certain types of redundant robotic chains (for details, see [2]).

On the other hand, in submanifold theory, one of the basic problems is to obtain simple basic relations between the extrinsic and intrinsic invariants of a submanifold. As we know, the main intrinsic invariants are the sectional curvature, Ricci curvature and the scalar curvature, and the main extrinsic invariant is the squared mean curvature of a manifold. An optimal way to obtain relationships between the intrinsic and extrinsic invariants of a submanifold is Chen invariants. The theory of Chen invariants was initiated by Chen in 1993. In [7], he established a sharp inequality for a submanifold in a real space form using the scalar curvature, sectional curvature and squared mean curvature $\|H\|^2$.

Chen invariants are one of the most popular topics in differential geometry and this theory have been extended to submanifolds of different kinds of manifolds (see [6, 9, 10, 12]). For instance, in [8], Chen studied Chen invariants for submanifolds in complex space forms and obtained a sharp inequality involving the intrinsic and extrinsic invariants. Since then, many papers have been published on this topic (see [11, 13, 25, 26]).

The paper is organized as follows: In Sec. 2, we recall some basic notions which are going to be needed. In Sec. 3, we establish some inequalities in order to give relationships between sectional curvature, scalar curvature and squared mean curvature for Lagrangian Riemannian submersions. Moreover, we obtain necessary and sufficient conditions for such submersions to be totally geodesic or totally umbilical. Also, we provide an example which satisfies inequality (38) in the present paper, such that the total space of such a submersion is Einstein. The last section is devoted to harmonicity. Here, we present a necessary and sufficient condition for which Lagrangian Riemannian submersion π is harmonic.

2. Preliminaries

In this section, we recall some basic notions about Riemannian submersions from [15]:

Definition 1. Let (M, g) and (B, g') be C^∞-Riemannian manifolds of dimension m and n, respectively. If a surjective C^∞-map

$$\pi : (M, g) \to (B, g')$$

has a maximal rank at any point of M, then we say that π is a C^∞-submersion.

For any $x \in B$, $\pi^{-1}(x)$ is closed r-dimensional submanifold of M, such that $r = m - n$. For any $p \in M$, we denote $\mathcal{V}_p = \ker \pi_{*p}$ and we obtain an integrable distribution $\mathcal{V}$.

Since each $\mathcal{V}_p$ coincides with the tangent space of $\pi^{-1}(x)$ at any point $\pi(p) = x$, one can see that an integrable distribution $\mathcal{V}$ corresponds to foliation of M, which is determined by the fibers of π. Indeed, $T_p\pi^{-1}(x)$ turns out to be an r-dimensional subspaces of $\mathcal{V}_p$ and it follows that $\mathcal{V}_p = T_p\pi^{-1}(x)$. Then, each $\mathcal{V}_p$ is called the vertical space at $p \in M$.

Denoting the complementary distribution of $\mathcal{V}$ by $\mathcal{H}$ which is determined by the Riemannian metric g. Then, we get

$$T_p(M) = \mathcal{V}_p \oplus \mathcal{H}_p. \tag{1}$$

where, $\mathcal{H}_p$ is called the horizontal space at any $p \in M$.

Definition 2. Let (M, g) and (B, g') be Riemannian manifolds and $\pi : (M, g) \to (B, g')$ be a submersion. At each point $p \in M$, if π_{*p} preserves the length of the horizontal vectors, then we say that a C^∞-submersion π is a Riemannian submersion.

Now, we give the following proposition presented in the basic properties of a Riemannian submersion from [15].

Proposition 3. *Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion, and denote by ∇ and ∇' the Levi-Civita connections of M and B, respectively. If X, Y are the basic vector fields, π-related to X', Y', one has*:

(1) $g(X, Y) = g'(X', Y') \circ \pi$;
(2) $h[X, Y]$ *is the basic vector field π-related to* $[X', Y']$;
(3) $h(\nabla_X Y)$ *is the basic vector field π-related to* $\nabla'_{X'} Y'$;
(4) *for any vertical vector field* $\mathcal{V}$, $[X, Y]$ *is the vertical.*

A Riemannian submersion $\pi : (M, g) \to (B, g')$ determines two $(1,2)$-tensor fields $\mathcal{T}$ and $\mathcal{A}$ on the base manifold M. As in [18], they are called the fundamental tensor fields or the invariants of Riemannian submersion π which are defined by

$$\mathcal{T}(E, F) = \mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF, \tag{2}$$

$$\mathcal{A}(E, F) = \mathcal{A}_E F = v\nabla_{hE} hF + h\nabla_{hE} vF, \tag{3}$$

such that the vertical and horizontal projections

$$v : \chi(M) \to \chi^v(M),$$

$$h : \chi(M) \to \chi^h(M),$$

where ∇ denote the Levi-Civita connection of M, for any $E, F \in \chi(M)$.

Using above equalities (2) and (3) and from [15], we obtain the followings:

Proposition 4. *Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion between Riemannian manifolds. Then, the tensor fields $\mathcal{T}$ and $\mathcal{A}$ hold the followings:*

(1) $\mathcal{T}_V W = \mathcal{T}_W V$,
(2) $\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2} v[X, Y]$,

for any $V, W \in \chi^v(M)$ and $X, Y \in \chi^h(M)$.

Remark 5. We here note that

(1) The restriction of A to $\chi^h(M) \times \chi^h(M)$ measures the integrability of horizontal distribution $\mathcal{H}$. Since, the vanishing of $\mathcal{A}$ means that the horizontal distribution $\mathcal{H}$ is integrable. Accordingly, the tensor field $\mathcal{A}$ is so-called the integrability tensor of Riemannian submersion.
(2) The restriction of $\mathcal{T}$ to $\chi^v(M) \times \chi^v(M)$ acts as the second fundamental form of any fiber. Since, the vanishing of $\mathcal{T}$ means that any fiber of Riemannian submersion is totally geodesic submanifold of M.

Using fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$, the Gauss–Codazzi type equations are given as follows:

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{4}$$

$$\nabla_V X = h\nabla_V X + \mathcal{T}_V X, \tag{5}$$

$$\nabla_X V = \mathcal{A}_X V + v\nabla_X V, \tag{6}$$

$$\nabla_X Y = h\nabla_X Y + \mathcal{A}_X Y \tag{7}$$

for any $V, W \in \chi^v(M)$ and $X, Y \in \chi^h(M)$.

Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion between Riemannian manifolds. Using the tensor field $\mathcal{T}$, we define the following bilinear mappings:

$$\begin{aligned}\mathcal{T}^h : \chi^v(M) \times \chi^v(M) &\to \chi^h(M)\\ (V,W) &\to \mathcal{T}^h_V W = h\nabla_V W,\end{aligned}$$

$$\begin{aligned}\mathcal{T}^v : \chi^v(M) \times \chi^h(M) &\to \chi^v(M)\\ (V,X) &\to \mathcal{T}^v_V X = v\nabla_V X,\end{aligned}$$

for any $V, W \in \chi^v(M)$ and $X \in \chi^h(M)$.
Let us denote the Riemannian curvature tensors of (M, g) and (B, g') by R and R', respectively. Then, we recall that

$$\pi_*(R^*(X,Y,Z)) = R'(\pi_* X, \pi_* Y, \pi_* Z)$$

for any horizontal vector fields of π.

Moreover, $\hat{R}$ stands for the Riemannian curvature of any fiber $(\pi^{-1}(x), g_x)$, then we get

$$R(U,V,F,W) = \hat{R}(U,V,F,W) - g(\mathcal{T}_U W, \mathcal{T}_V F) + g(\mathcal{T}_V W, \mathcal{T}_U F), \tag{8}$$

$$\begin{aligned}R(X,Y,Z,H) &= R^*(X,Y,Z,H) + 2g(\mathcal{A}_X Y, \mathcal{A}_Z H)\\ &\quad - g(\mathcal{A}_Y Z, \mathcal{A}_X H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H),\end{aligned} \tag{9}$$

$$\begin{aligned}R(X,V,Y,W) &= -g((\nabla_X \mathcal{T})(V,W), Y) - g((\nabla_V \mathcal{A})(X,Y), W)\\ &\quad + g(\mathcal{T}_V X, \mathcal{T}_W Y) - g(\mathcal{A}_X V, \mathcal{A}_Y W),\end{aligned} \tag{10}$$

for any $U, V, W, F \in \chi^v(M)$ and $X, Y, Z, H \in \chi^h(M)$.

Let $\{U, V\}$ be an orthonormal basis of the vertical 2-plane. From (8), one has the following equality between the sectional curvature on M and the fiber through p, as follows:

$$K(U,V) = \hat{K}(U,V) - \|\mathcal{T}_U V\|^2 + g(\mathcal{T}_U U, \mathcal{T}_V V). \tag{11}$$

Let $\{X, Y\}$ be an orthonormal basis of the horizontal 2-plane, denoting the sectional curvature in (B, g') by $K'(\pi_* X, \pi_* Y)$. Considering (9), we have

$$K(X,Y) = K'(\pi_* X, \pi_* Y) + 3\|\mathcal{A}_X Y\|^2. \tag{12}$$

Finally, for any unit vector fields $X \in \chi^h(M), V \in \chi^v(M)$, using (10), one has

$$K(X,V) = -g((\nabla_X \mathcal{T})_V V, X) + \|\mathcal{T}_V X\|^2 - \|\mathcal{A}_X V\|^2, \tag{13}$$

such that $\{X_i, V_j\}_{1\le i\le n; 1\le j\le r}$ is called π-adapted a locally orthonormal frame on M (see [15]).

The scalar curvature $\tau(p)$ at any $p \in M$ is given as

$$\tau(p) = \sum_{1\le i<j\le r} K(U_i, U_j) + \sum_{1\le i<j\le n} K(X_i, X_j) + \sum_{j=1}^{r}\sum_{i=1}^{n} K(U_j, X_i). \tag{14}$$

where K stands for the sectional curvature of M.

On the other hand, the mean curvature vector field $H(p)$ on any fiber of Riemannian submersion π as follows:

$$N = rH, \tag{15}$$

such that

$$N = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j \tag{16}$$

and r denotes the dimension of any fiber of π and $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis on vertical distribution. We here note that the horizontal vector field N vanishes if and only if any fiber of Riemannian submersion π is minimal submanifold on M.

Let us recall that if $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis of vertical space, then we get

$$g(\nabla_E N, X) = \sum_{j=1}^{r} g((\nabla_E \mathcal{T})_{U_j} U_j, X). \tag{17}$$

for any $E \in \chi(M)$ and $X \in \chi^h(M)$ (see [15]).

Denoting the horizontal divergence of any vector field X on $\chi^h(M)$ by $\check{\delta}(X)$ and given by

$$\check{\delta}(X) = \sum_{i=1}^{n} g(\nabla_{X_i} X, X_i), \tag{18}$$

where $\{X_1, X_2, \ldots, X_n\}$ is an orthonormal basis of horizontal space $\chi^h(M)$. Hence, we get

$$\check{\delta}(N) = \sum_{i=1}^{n} \sum_{j=1}^{r} g((\nabla_{X_i} \mathcal{T})_{U_j}, U_j, X_i). \tag{19}$$

For details, we refer ([5, pp. 243]).

3. Some Inequalities for Lagrangian Riemannian Submersions

First, we recall the following notions of [4]:

Definition 6. Let M be an m-dimensional manifold and $\mathcal{F} = \{L_t\}_{t \in I}$ be a family of connected subsets of M. Suppose that $\mathcal{F}$ is a partition of M, that is one has

$$M = \bigcup_{t \in I} L_t \quad \text{and} \quad L_t \cap L_s = \emptyset, \quad \text{for } t \neq s.$$

Next, we consider a positive integer $n < m$ and a local chart $(\mathcal{U}, \varphi)$ on M. Then, we say that $(\mathcal{U}, \varphi)$ is an n-foliated chart, if whenever $L_t \cap \mathcal{U} \neq 0$, for some $t \in I$, then each connected component of $L_t \cap \mathcal{U}$ is mapped by φ onto n-plaque of $\mathbb{R}^m$. An n-foliated atlas associated to $\mathcal{F}$ on M is a collection of n-foliated charts whose domains cover M. Then, we say that the partition $\mathcal{F}$ on M is a foliation of dimension n, if there exits on M a maximal n-foliated atlas associated with $\mathcal{F}$. We also say that $(M, \mathcal{F})$ is an n-foliated manifold, and $\mathcal{F}$ is an n-foliation of M.

A foliation $\mathcal{F}$ corresponds to a decomposition of a manifold into a union of connected submanifolds of the same dimension called leaves.

The Riemannian submersion π defines a foliation of M which is given in Definition 2. Thus, the total space of fiber bundle has n-foliation whose leaves are the components of fibers of π. It follows that the tangent distribution to the foliation $\mathcal{F}$ determined by the fibers of Riemannian submersion π is called the vertical distribution and it is denoted by $\mathcal{V}$.

On the other hand, we give here the definition of [28] as follows:

Let M be a real differentiable manifold. A tensor field $\mathfrak{J}$ on M is called an almost complex structure on M, if at every point x of M, $\mathfrak{J}$ is an endomorphism of the tangent space $T_x(M)$ such that $\mathfrak{J}^2 = -I$. A manifold M with a fixed almost complex structure $\mathfrak{J}$ is called an almost complex manifold. Also, M is called a complex manifold if M admits a linear connection ∇ such that $\nabla\mathfrak{J} = 0$ and $T = 0$, where T denote the torsion of ∇.

Suppose that M is an almost complex manifold with almost complex structure $\mathfrak{J}$. A Hermitian metric on M is a Riemannian metric g such that

$$g(\mathfrak{J}X, \mathfrak{J}X) = g(X, Y), \tag{20}$$

for any X and Y on M. An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold and a complex manifold with a Hermitian metric is called a Hermitian manifold.

Now, we may give the following notions:

Let M be a complex n-dimensional almost Hermitian manifold with almost Hermitian metric g, almost complex structure $\mathfrak{J}$ and B be a Riemannian manifold with Riemannian metric g'. Suppose that there exists a Riemannian submersion $\pi : (M, \mathfrak{J}, g) \to (B, g')$, such that the vertical distribution $\mathcal{V}$ is anti-invariant with respect to $\mathfrak{J}$, i.e. $\mathfrak{J}\mathcal{V} \subseteq \mathcal{H}$. Then, the Riemannian submersion π is called an anti-invariant Riemannian submersion (for details, [19]).

Furthermore, suppose that M is a complex n-dimensional almost Hermitian manifold with almost Hermitian metric g, almost complex structure $\mathfrak{J}$ and B is a Riemannian manifold with Riemannian metric g'. Let $\pi : (M, \mathfrak{J}, g) \to (B, g')$ be an anti-invariant Riemannian submersion. Then, we say that π is a Lagrangian Riemannian submersion, if the dimension of the vertical distribution $\mathcal{V}$ is equal to the dimension of the horizontal distribution $\mathcal{H}$, i.e. $\dim(\ker \pi_*) = \dim(\ker \pi_*)^{\perp}$. In this case, an almost complex structure $\mathfrak{J}$ of M reverses the vertical and horizontal distributions, i.e. $\mathfrak{J}\mathcal{V} = \mathcal{H}$ and $\mathfrak{J}\mathcal{H} = \mathcal{V}$ (see [23]).

On the other hand, from [4], we present the relationship between the symplectic geometry and complex geometry. First, suppose that $(M, \mathfrak{J}, g)$ is an almost Kähler manifold with fundamental 2-form Ω which is defined by

$$\Omega(E,F) = g(E,\mathfrak{J}F),$$

for any $E,F \in \chi(M)$. As Ω is closed, we say that (M,Ω) is a symplectic manifold. We infer here that any symplectic manifold (M,Ω) admits a Riemannian metric g and an almost complex structure $\mathfrak{J}$ such that $(M,\mathfrak{J},g)$ is an almost Kähler manifold. Then, it is called $(g,\mathfrak{J})$ the associated almost Kähler structure to the symplectic structure defined by Ω on M. Hence it is important to say that a smooth manifold admits a symplectic structure if and only if it admits an almost Kähler structure.

Remark 7. Let (M,Ω) be a $2n$-dimensional symplectic manifold and $\mathcal{F}$ be a foliation if all fiber of $\mathcal{F}$ is a Lagrangian submanifold of M. *Since*, $\mathcal{V}$ *is the tangent distribution to* $\mathcal{F}$ *and any fibers of* $\mathcal{V}$ are Lagrange submanifolds of TM, then $\mathcal{F}$ is called a Lagrangian foliation (see [4]).

Remark 8. Let $\pi : (M,\mathfrak{J},g) \to (B,g')$ be a Lagrangian Riemannian submersion. Then, the followings are hold:

(1) $\mathcal{F}$ is Lagrangian foliation,
(2) M is the total space of π admits a symplectic structure.

Firstly, we begin to this section with the following lemma from [25]:

Lemma 9. *If* $a_1, a_2, \ldots, a_n (n > 1)$ *are real numbers, then*

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i=1}^{n}(a_i^2), \tag{21}$$

with equality holding if and only if $a_1 = a_2 = \cdots = a_n$.

The following equalities give us relationships between the scalar curvatures of the total space M, the vertical and horizontal spaces of Lagrangian Riemannian submersion π.

Lemma 10. *Let* $(M,\mathfrak{J},g) \to (B,g')$ *be a Lagrangian Riemannian submersion from a Hermitian manifold to Riemannian manifold. Then, we have*

$$2\tau^*(p) = 2\hat{\tau}(p) - \|\mathcal{T}^h_{\mathcal{V}\times\mathcal{V}}\|^2 + n^2\|H\|^2, \tag{22}$$

$$2\tau(p) = 2\hat{\tau}(p) + 2\tau^*(p) - \check{\delta}(N) + n^2\|H\|^2, \tag{23}$$

where

$$\hat{\tau}(p) = \sum_{1\leq i,j\leq n} K(U_i,U_j),$$

$$\tau^*(p) = \sum_{1\leq i,j\leq n} K(\mathfrak{J}U_i,\mathfrak{J}U_j),$$

are the vertical and the horizontal n*-plane section scalar curvatures of* $T_p(M)$*, respectively.*

Proof. Since M is Hermitian manifold, it is known that

$$\sum_{i,j=1}^{n} R(U_i, U_j, U_j, U_i) = \sum_{i,j=1}^{n} R(\mathfrak{J}U_i, \mathfrak{J}U_j, \mathfrak{J}U_j, \mathfrak{J}U_i). \tag{24}$$

Here R denotes the Riemannian curvature tensor of M and using Gauss–Codazzi type equations for equality (24), one has

$$\sum_{i,j=1}^{n} \hat{R}(U_i, U_j, U_j, U_i) + \sum_{i,j,k=1}^{n} ((\mathcal{T}_{ij}^k)^2 - \mathcal{T}_{ii}^k \mathcal{T}_{jj}^k) = \sum_{i,j=1}^{n} R^*(\mathfrak{J}U_i, \mathfrak{J}U_j, \mathfrak{J}U_j, \mathfrak{J}U_i). \tag{25}$$

Considering equality (25), the required relation (22) is obtained.

On the other hand, in order to prove equality (23), we consider the scalar curvature for Lagrangian Riemannian submersion π. Then,

$$\tau(p) = \frac{1}{2}\sum_{i,j=1}^{n} R(U_i, U_j, U_j, U_i) + \frac{1}{2}\sum_{i,j=1}^{n} R(\mathfrak{J}U_i, \mathfrak{J}U_j, \mathfrak{J}U_j, \mathfrak{J}U_i) + \frac{1}{2}\sum_{i,j=1}^{n} R(U_i, \mathfrak{J}U_j, \mathfrak{J}U_j, U_i),$$

where $\{U_i\}_{1\le i\le n}$ is an orthonormal basis of the vertical space $\chi^v(M)$. Considering Gauss–Codazzi equations (8), (9) and (10) for Riemannian submersions, one has

$$\begin{aligned} 2\tau(p) = {} & \sum_{i,j=1}^{n} \hat{R}(U_i, U_j, U_j, U_i) - \sum_{i,j,k=1}^{n} (\mathcal{T}_{ij}^k)^2 + \sum_{i,j,k=1}^{n} \mathcal{T}_{ii}^k \mathcal{T}_{jj}^k \\ & + \sum_{i,j=1}^{n} R^*(\mathfrak{J}U_i, \mathfrak{J}U_j, \mathfrak{J}U_j, \mathfrak{J}U_i) - \sum_{i,j=1}^{n} g((\nabla_{\mathfrak{J}U_i}\mathcal{T})_{U_j} U_j, \mathfrak{J}U_i) + \sum_{i,j,k=1}^{n} (\mathcal{T}_{ij}^k)^2. \end{aligned} \tag{26}$$

Putting the relation (15) and horizontal divergence as (19) in (26). □

We may characterize the above notions as follows:

Theorem 11. *Let $\pi : (M, \mathfrak{J}, g) \to (B, g')$ be Lagrangian Riemannian submersion from a Hermitian manifold to Riemannian manifold, such that the total space of π admits an n-foliation $\mathcal{F}$. Then, we have*

$$2\hat{\tau}(p) - n(1-n)\|H\|^2 \ge 2\tau^*(p), \tag{27}$$

$$2\tau(p) \le 2\hat{\tau}(p) + 2\tau^*(p) - n(1-n)\|H\|^2 + \|\mathcal{T}_{V\times H}^v\|^2 - \check{\delta}(N). \tag{28}$$

Equality case of (27) *and* (28) *holds if and only if one of the following is true*:

(1) *The Lagrangian Riemannian submersion π has totally umbilical fibers,*
(2) *The foliation $\mathcal{F}$ is totally umbilical foliation.*

We here note that, if $(M, \mathfrak{J}, g)$ is a Kähler manifold,

$$\nabla_U \mathfrak{J} V = \mathfrak{J} \nabla_U V, \tag{29}$$

then, one has

$$\mathcal{T}_U \mathfrak{J} V = \mathfrak{J} \mathcal{T}_U V, \tag{30}$$

for any $U, V \in \chi^v(M)$. Using the relation (30), we get

$$\|\mathcal{T}^h_{V\times V}\|^2 = \|\mathcal{T}^v_{V\times H}\|^2. \tag{31}$$

Using the Kähler condition, we have the following lemma:

Lemma 12. *Let $\pi : (M, \mathfrak{J}, g) \to (B, g')$ be a Lagrangian Riemannian submersion from Kähler manifold to Riemannian manifold. Then, one has*

$$2\tau(p) = 4\hat{\tau}(p) - \|\mathcal{T}^h_{V\times V}\|^2 + 2n^2\|H\|^2 - \check{\delta}(N). \tag{32}$$

$$2\tau(p) = 4\tau^*(p) + \|\mathcal{T}^v_{V\times H}\|^2 - \check{\delta}(N). \tag{33}$$

Proof. If we use the scalar curvature for Lagrangian Riemannian submersion, we get

$$\tau(p) = \sum_{i,j=1}^{n} R(U_i, U_j, U_j, U_i) + \frac{1}{2}\sum_{i,j=1}^{n} R(\mathfrak{J}U_i, U_j, U_j, \mathfrak{J}U_i).$$

Putting Gauss–Codazzi type equations (8) and (10) in above equality (32), one has

$$2\tau(p) = 2\sum_{i,j=1}^{n} \hat{R}(U_i, U_j, U_j, U_i) - \sum_{i,j,k=1}^{n} (\mathcal{T}^k_{ij})^2 + 2\sum_{i,j,k=1}^{n} \mathcal{T}^k_{ii}\mathcal{T}^k_{jj}$$
$$- \sum_{i,j=1}^{n} g((\nabla_{\mathfrak{J}U_i}\mathcal{T})(U_j, U_j), \mathfrak{J}U_i).$$

Then, the required equality (32) is obtained. Also, using Gauss–Codazi type equations for Riemannian submersions, it is clear that

$$\tau(p) = \sum_{i,j=1}^{n} R(\mathfrak{J}U_i, \mathfrak{J}U_j, \mathfrak{J}U_j, \mathfrak{J}U_i) + \frac{1}{2}\sum_{i,j=1}^{n} R(\mathfrak{J}U_i, U_j, U_j, \mathfrak{J}U_i).$$

Then, it follows that

$$2\tau(p) = 2\sum_{i,j=1}^{n} R^*(\mathfrak{J}U_i, \mathfrak{J}U_j, \mathfrak{J}U_j, \mathfrak{J}U_i) - \sum_{i,j=1}^{n} g((\nabla_{\mathfrak{J}U_i}\mathcal{T})(U_j, U_j), \mathfrak{J}U_i)$$
$$+ \sum_{i,j,k=1}^{n} (\mathcal{T}^k_{ij})^2,$$

which proves the required equality (33). □

Corollary 13. *Let $\pi : (M, \mathfrak{J}, g) \to (B, g')$ be a Lagrangian Riemannian submersion with totally geodesic fibers from Kähler manifold to Riemannian manifold. Consid-*

ering relations (32) *and* (33), *we conclude that the scalar curvatures of the vertical space and horizontal space are the same.*

We here give as following characterizations using above equalities (32) and (33):

Theorem 14. *Let* $\pi : (M, \mathfrak{J}, g) \to (B, g')$ *be a Lagrangian Riemannian submersion from Kähler manifold to Riemannian manifold, such that the total space of* π *admits an* n*-foliation* $\mathcal{F}$. *Then, we get*

$$2\tau(p) \leq 4\hat{\tau}(p) + 2n^2\|H\|^2 - \check{\delta}(N), \tag{34}$$

$$2\tau(p) \geq 4\tau^*(p) - \check{\delta}(N). \tag{35}$$

Equality case of (34) *and* (35) *holds for any* $p \in M$ *if and only if any fiber through* p *of* π *is totally geodesic submanifold of* M, *or equivalently the foliation* $\mathcal{F}$ *is totally geodesic foliation. In additionally, we have*

$$2\tau(p) \leq 4\hat{\tau}(p) + n(2n-1)\|H\|^2 - \check{\delta}(N), \tag{36}$$

$$2\tau(p) \geq 4\tau^*(p) + n^2\|H\|^2 - \check{\delta}(N). \tag{37}$$

Equality case of (36) *and* (37) *holds for any* $p \in M$ *if and only if the Lagrangian Riemannian submersion* π *has totally umbilical fibers, or equivalently the foliation* $\mathcal{F}$ *is totally umbilical foliation.*

Now, we recall the following lemma from [25]:

Lemma 15. *If* $n > k \geq 2$ *and* $a_1, a_2, \ldots, a_n, a$ *are real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n+k-1)\left(\sum_{i=1}^{n}(a_i^2 + a)\right),$$

then

$$2\sum_{1\leq i<j\leq k} a_i a_j \geq a,$$

with equality holding if and only if

$$a_1 + a_2 = a_3 = \cdots = a_k = a_{k+1} = \cdots = a_n.$$

Using Lemma 15, we get the following theorem:

Theorem 16. *Let* $\pi : (M, \mathfrak{J}, g) \to (B, g')$ *be a Lagrangian Riemannian submersion from Hermitian manifold to Riemannian manifold. Then, one has*:

$$\tau(p) - K_P \leq \hat{\tau}(p) + \tau^*(p) - \hat{K}_P + \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}\|\mathcal{T}^v_{V\times H}\|^2 - \frac{1}{2}\check{\delta}(N). \tag{38}$$

Equality case of (38) *holds for any* $p \in M$ *if and only if matrices of* $\mathcal{T}^h$ *at* $p \in M$ *have the following forms*:

$$\mathcal{A}_{\mathfrak{J}U_1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}, \quad \lambda = a + b$$

and

$$\mathcal{A}_{\mathfrak{J}U_k} = \begin{pmatrix} c_k & d_k & \dots & 0 \\ d_k & -c_k & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad 2 \leq k \leq n.$$

Proof. If we put

$$w = 2\hat{\tau}(p) + 2\tau^*(p) - 2\tau(p) + \|\mathcal{T}^v_{V\times H}\|^2 + \frac{n^2(n-2)}{n-1}\|H\|^2 - \check{\delta}(N). \tag{39}$$

Thus, from (30),

$$w - \frac{n^2}{n-1}\|H\|^2 + \|\mathcal{T}^h_{V\times V}\|^2 = 0$$

is obtained. Also, let $P = \{U_1, U_2\}$ be a 2-plane section of $\chi^v(M)$. Using relation (8),

$$R(U_1,U_2,U_2,U_1) = \hat{R}(U_1,U_2,U_2,U_1) + \mathcal{T}^1_{11}\mathcal{T}^1_{22} + \sum_{k=2}^{n}(\mathcal{T}^k_{11}\mathcal{T}^k_{22} - (\mathcal{T}^k_{12})^2). \tag{40}$$

Moreover, from Lemma 15,

$$\mathcal{T}^1_{11}\mathcal{T}^1_{22} \geq -\frac{w}{2} + \frac{1}{2}\sum_{i\neq j=1}^{n}(\mathcal{T}^1_{ij})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=2}^{n}(\mathcal{T}^k_{ij})^2 \tag{41}$$

can be written. Thus,

$$\begin{aligned} R(U_1,U_2,U_2,U_1) \geq{}& \hat{R}(U_1,U_2,U_2,U_1) - \frac{w}{2} + \frac{1}{2}\sum_{k=2}^{n}(\mathcal{T}^k_{11} + \mathcal{T}^k_{22})^2 \\ &+ \frac{1}{2}\sum_{k=2}^{n}\sum_{i,j>2}(\mathcal{T}^k_{ij})^2 + \sum_{k=1}^{n}\sum_{j>2}((\mathcal{T}^k_{1j})^2 + (\mathcal{T}^k_{2j})^2). \end{aligned} \tag{42}$$

Hence,

$$R(U_1,U_2,U_2,U_1) \leq \hat{R}(U_1,U_2,U_2,U_1) - \frac{w}{2}$$

which implies

$$K_P \leq \hat{K}_P - \frac{w}{2} \tag{43}$$

is obtained. Using (39) and (41), we get

$$R(U_1,U_2,U_2,U_1) \geq \hat{R}(U_1,U_2,U_2,U_1) - \hat{\tau}(p) - \tau^*(p) + \tau(p) - \frac{n^2(n-2)}{2(n-1)}\|H\|^2$$

$$-\frac{1}{2}\|\mathcal{T}^v_{V\times H}\|^2 + \frac{1}{2}\check{\delta}(N),$$

which represents the proof of inequality (38). Moreover, equality case of (38) holds for any $p \in M$ if and only if

$$\sum_{k=2}^{n}(\mathcal{T}^k_{11} + \mathcal{T}^k_{22}) = 0, \quad \sum_{k=1}^{n}\sum_{j>2}^{n}((\mathcal{T}^k_{1j})^2 + (\mathcal{T}^k_{2j})^2) = 0,$$

$$\sum_{k=2}^{n}\sum_{j>2}^{n}(\mathcal{T}^k_{ij})^2 = 0, \quad \mathcal{T}^1_{11} + \mathcal{T}^1_{22} = \mathcal{T}^1_{33} = \cdots = \mathcal{T}^1_{nn},$$

which follows that the matrices of tensor $\mathcal{T}^h$ at any $p \in M$ take the desired form.

□

Proposition 17. *Let $\pi : (M, \mathfrak{J}, g) \to (B, g')$ be a Lagrangian Riemannian submersion with totally geodesic from a Hermitian manifold to Riemannian manifold. Then, the total space $(M, \mathfrak{J}, g)$ is Einstein if and only if the Riemannian manifold B and fibers of such a submersion are Einstein* (*for details*, *see* [17]).

Now, we provide some classes of examples which satisfy our main inequality (38) as follows:

Example 18. A Lagrangian Riemannian submersion $\pi : (M, \mathfrak{J}, g) \to (B, g')$ with totally geodesic from a Hermitian manifold to Riemannian manifold, such that the total space M is an Einstein. Considering Proposition 17, one can see that such a submersion $\pi : M \to B$ satisfy the inequality (38).

4. Lagrangian Riemannian Submersions and Their Harmonicity

In this section, we study the harmonicity of Lagrangian Riemannian submersions. As a tool, we use here the tension field and provide a necessary and sufficient condition for which Lagrangian Riemannian submersion π is harmonic on M.

We recall now the following notion of [3] and [14]:

Definition 19. Let $\pi : (M^m, g) \to (B^n, g')$ be a smooth map between Riemannian manifolds and let ∇ and $\nabla^{\pi^{-1}TB}$ denote the Levi-Civita connection on M and the pull-back connection, respectively. Then, π is harmonic if its tension field $\sigma(\pi)$ vanishes identically, that is,

$$\sigma(\pi) = \mathrm{trace}_g(\nabla_{\cdot}\pi_{*\cdot}) = \sum_{i=1}^{m}(\nabla\pi_*)(e_i, e_i) = 0, \tag{44}$$

where $\{e_i\}_{i=1,\ldots,m}$ is an orthonormal basis on M and the second fundamental form $\nabla\pi_*$ of π is defined by

$$(\nabla\pi_*)(E,F) = \nabla_E^{\pi^{-1}TB}\pi_*F - \pi_*(\nabla_E F), \quad \text{for any } E, F \in \chi(M).$$

Theorem 20. *Let $\pi : (M, \mathfrak{J}, g) \to (B, g')$ be a Lagrangian Riemannian submersion from Kähler manifold to Riemannian manifold. Then, π is harmonic if and only if any fiber through p of π is minimal.*

Proof. Using the tension field, we get

$$\begin{aligned}\sigma(\pi) &= \mathrm{trace}_g(\nabla_{\boldsymbol{.}}\pi_{*\boldsymbol{.}}) = \sum_{i=1}^{2n}(\nabla\pi_*)(E_i,E_i) = \sum_{i=1}^{2n}\{\nabla_{E_i}^{\pi^{-1}TB}\pi_*E_i - \pi_*(\nabla_{E_i}E_i)\}\\ &= \sum_{j=1}^{n}\{\nabla_{U_j}^{\pi^{-1}TB}\pi_*U_j - \pi_*(\nabla_{U_j}U_j)\} + \sum_{i=1}^{n}\{\nabla_{\mathfrak{J}U_i}^{\pi^{-1}TB}\pi_*\mathfrak{J}U_i - \pi_*(\nabla_{\mathfrak{J}U_i}\mathfrak{J}U_i)\}\\ &\quad + \sum_{i,j=1}^{n}\{\nabla_{\mathfrak{J}U_i}^{\pi^{-1}TB}\pi_*U_j - \pi_*(\nabla_{\mathfrak{J}U_i}U_j)\} + \sum_{i,j=1}^{n}\{\nabla_{U_j}^{\pi^{-1}TB}\pi_*\mathfrak{J}U_i - \pi_*(\nabla_{U_j}\mathfrak{J}U_i)\},\end{aligned} \tag{45}$$

where $\{E_1,\ldots,E_{2n}\}$, $\{U_1,\ldots,U_n\}$ and $\{\mathfrak{J}U_1,\ldots,\mathfrak{J}U_n\}$ denote the orthonormal basis of total space M, the vertical and horizontal spaces, respectively.

Considering Gauss–Codazzi type equations for Riemannian submersions, that is equalities (4)–(7) and the Kähler condition for M gives:

$$\begin{aligned}\sigma(\pi) &= \sum_{j=1}^{n}\nabla_{U_j}^{\pi^{-1}TB}\pi_*U_j + \sum_{i=1}^{n}\nabla_{\mathfrak{J}U_i}^{\pi^{-1}TB}\pi_*(\mathfrak{J}U_i) + \sum_{i,j=1}^{n}\nabla_{\mathfrak{J}U_i}^{\pi^{-1}TB}U_j\\ &\quad + \sum_{i,j=1}^{n}\nabla_{U_j}^{\pi^{-1}TB}\mathfrak{J}U_i - \sum_{j=1}^{n}\pi_*(\mathcal{T}_{U_j}U_j) - \sum_{i=1}^{n}\pi_*(h\nabla_{\mathfrak{J}U_i}\mathfrak{J}U_i)\\ &\quad - \sum_{i,j=1}^{n}\pi_*(h\nabla_{U_j}\mathfrak{J}U_i) - \sum_{i,j=1}^{n}\pi_*(\mathcal{A}_{\mathfrak{J}U_i}U_j).\end{aligned}$$

Using equalities (5)–(7), it follows that

$$\sigma(\pi) = -\sum_{j=1}^{n}\pi_*(\mathcal{T}_{U_j}U_j). \tag{46}$$

Equality (46) means that the proof is completed. □

Now, we give our main theorem as follows:

Theorem 21. *Let $\pi : (M, \mathfrak{J}, g) \to (B, g')$ be a Lagrangian Riemannian submersion from Kähler manifold to Riemannian manifold, such that the total space of π admits* *an n-foliation $\mathcal{F}$. Then, the following conditions are equivalent to each other*:

(i) *π is harmonic*,
(ii) *the vertical distribution $\mathcal{V}$ is minimal*,
(iii) *the foliation $\mathcal{F}$ is minimal foliation.*

Proof. (i) $\Rightarrow$ (ii) Suppose that π is harmonic. Then, the tension field $\sigma(\pi)$ is vanishes. Using Theorem 20, it is clear that any fiber of Lagrangian Riemannian submersion π is minimal. This means the vertical distribution $\mathcal{V}$ is minimal. (ii) $\Rightarrow$ (iii) Let the vertical distribution $\mathcal{V}$ be a minimal. It is known that, the Lagrangian Riemannian submersion π defines a foliation $\mathcal{F}$ whose leaves are components of fibers of π. It follows that the tangent distribution to the foliation $\mathcal{F}$ determined by the fibers of Riemannian submersion π. Hence, (iii) is clear. (iii) $\Rightarrow$ (i) If the foliation $\mathcal{F}$ is minimal, any fiber of Lagrangian Riemannian submersion π is minimal. Then, from (16) and (46), (iii) is desired. □

Acknowledgments

The authors are grateful to referees for their valuable comments and suggestions.

References

[1] P. Alegre, B. Y. Chen and M. I. Munteanu, Riemannian submersions, δ-invariants and optimal inequality, *Ann. Global Anal. Geom.* **42** (2012) 317–331.

[2] C. Altafini, Redundant robotic chains on Riemannian submersions, *IEEE. Tran. Robot. Autom.* **20** (2004) 335–340.

[3] C.-L. Bejan and Ş. Eken, Conformality on semi-Riemannian manifolds, *Mediterr. J. Math.* **13** (2016) 2185–2198.

[4] A. Bejancu and H. R. Farran, *Foliations and Geometric Structures* (Springer, The Netherlands, 2006).

[5] A. L. Besse, *Einstein Manifolds* (Springer-Verlag, Berlin, 1987).

[6] B. Y. Chen and M. Okumura, Scalar curvature, inequality and submanifold, *Proc. Amer. Math. Soc.* **3** (1973) 605–608.

[7] B. Y. Chen, Some pinching and classification theorems for minimal submanifolds, *Archiv. Math.* **60** (1993) 568–578.

[8] B. Y. Chen, A general inequality for submanifolds in complex space forms and its applications, *Archiv. Math.* **67** (1996) 519–528.

[9] B. Y. Chen, Mean curvature and shape operator of isometric immersions in real space form, *Glasgow Math. J.* **38** (1996) 87–97.

[10] B. Y. Chen, Riemannian geometry of Lagrangian submanifolds, *Taiwanese J. Math.* **5** (2001) 681–723.

[11] B. Y. Chen, Riemannian submersions, minimal immersions and cohomology class, *Proc. Japan Acad.* **81** (2005) 162–167.

[12] B. Y. Chen and S. Wei, p-Harmonic morhisms, cohomology classes and submersions, *Tamkang J. Math.* **40** (2009) 377–382.

[13] Ş. Eken, M. Gülbahar and E. Kılıç, Some inequalities for Riemannian submersions, *Scientific Annals of the Alexandru Ioan Cuza University of Iasi*, in press.

[14] J. Eells and H. J. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964) 109–160.

[15] M. Falcitelli, S. Ianus and A. M. Pastore, *Riemannian Submersions and Related Topics* (World Scientific Publishing, 2004).

[16] A. Gray, Pseudo-Riemannian almost product manifolds and submersions, *J. Math. Metch.* **16** (1967) 715–737.

[17] J. Lee, J. Park, B. Şahin and D. Song, Einstein conditions for the base space of anti-invariant Riemannian submersions and its Clairaut submersions, *Taiwanese J. Math.* **19**(4) (2015) 1145–1160.

[18] B. O'Neill, The fundamental equations of a Riemannian submersions, *Mich. Math. J.* **13** (1966) 459–469.

[19] B. Şahin, Anti-invariant Riemannian submersions from almost Hermitian manifolds, *Cent. Eur. J. Math.* **8** (2010) 437–447.

[20] B. Şahin, Slant submersions from almost Hermitian submersions, *Bull. Math. Soc. Sci. Math. Roumanie Tome* **54** (2011) 93–105.

[21] B. Şahin, Semi-invariant Riemannian submersions from almost Hermitian manifolds, *Canadian Math. Bull.* **56** (2013) 173–187.

[22] B. Şahin, Riemannian submersions from almost Hermitian manifolds, *Taiwanese J. Math.* **17** (2013) 629–659.

[23] H. M. Taştan, On Lagrangian submersions, *Hacet. J. Math. Stat.* **43** (2014) 993–1000.

[24] H. M. Taştan, B. Şahin and Ş. Yanan, Hemi-slant submersions, *Mediterr. J. Math.* **13** (2016) 2171–2184.

[25] M. M. Tripathi, Certain basic inequalities for submanifolds in (κ, μ)-space, in *Recent Advances in Riemannian and Lorentzian Geometries*, eds. K. L. Duggal and R. Sharma, Amer. Math. Soc., Providence, RI, Contemp. Math., Vol. 337 (Baltimore, MD, 2003), pp. 187–202.

[26] G. E. Vîlcu, On Chen invariants and inequalities in quaternionic geometry, *J. Inequalities Appl.* **2013** (2013) 66.

[27] B. Watson, Almost Hermitian submersions, *J. Differential Geometry* **11** (1976) 147–165.

[28] K. Yano and M. Kon, *Structures on Manifolds* (World Scientific Publishing, Singapore, 1984).