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Some Inequalities for Riemannian Submersions

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Abstract In this paper, we start by studying the scalar curvature of two Riemannian manifolds admitting a Riemannian submersion. We establish a series of inequalities for Riemannian submersions. By using these inequalities, we derive several characterizations for Riemannian submersions. We show that the necessary conditions for a Riemannian submersion to be harmonic is to either have totally geodesic fibres or integrable horizontal distribution.

Keywords Curvature · Riemannian submersion · Harmonic map

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1 Introduction

The theory of smooth maps between Riemannian manifolds plays a significant role in differential geometry since these maps are useful to compare various geometric properties between Riemannian manifolds. The best known of such maps are isometric immersions and Riemannian submersions. The origins of the theory of isometric immersions and of the submanifold geometry can be traced back to the 1827 seminal paper written by C. F. Gauss on surfaces in the three dimensional Euclidean space in [12]. Pursuing an idea of M. Janet in [14] and E. Cartan in [3], J. F. Nash [16] proved a theorem generated as a revolution for Riemannian manifolds in 1956 that

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every Riemannian n-manifold can be isometrically embedded in any small portion of a Euclidean spaces E^m with $m = \frac{n}{2}(n+1)(3n+11)$. From the Nash's embedding theorem, one of basic problems in submanifolds theory is to find simple relations between the intrinsic invariants and the extrinsic invariants of a Riemannian submanifold.

In 1973, B.-Y. Chen and M. Okumura [10] showed that if M is a n-dimensional submanifold of a real space form of constant curvature c such that the scalar curvature τ and the squared second fundamental form $|\sigma|^2$ satisfy the inequality

$$2\tau \geq (n-2)\,|\sigma|^2 + (n-2)(n-1)c \tag{1.1}$$

at a point $p \in M$, then the sectional curvatures of M are nonnegative at p.

In 1996, B.-Y. Chen [5] obtained a substantial inequality for a $n-$ dimensional submanifold of a real space form $R^m(c)$, which involves the scalar curvature and the squared mean curvature. He proved that at every point $p \in M$, the scalar curvature τ and the squared mean curvature $|H|^2$ of M satisfy

$$\tau \leq \frac{1}{2}\,n(n-1)\,|H|^2 + \frac{1}{2}\,n(n-1)c, \tag{1.2}$$

with equality holding if and only if p is a totally umbilical point.

In literature, these types inequalities are known as Chen-like inequalities or Chen inequalities.

On the other hand, the theory of Riemannian submersion goes back to five decades ago, when B. O'Neill [17] and A. Gray [13] formulated the basis of such theory, which has hugely developed in the three decades. Also, B. O'Neill [17] introduced and studied two tensor fields for Riemannian submersions corresponding to second fundamental form and shape operator in isometric immersions which are known as O'Neill tensors. Then, he gave some relations involving sectional curvatures of two manifolds admitting a Riemannian submersion. Later, Riemannian submersions have been studied widely by various mathematicians, motivated be the importance and the applications of these maps in mathematical physics. In particular, in Kaluza-Klein theory, Riemannian submersions are general class of solutions of the model can be expressed in terms of harmonic maps satisfyng Einstein equations. One can see details of this result in [11].

In [9], B.-Y. Chen established an inequality for Riemannian submersions as follows:

Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibres. Then, for any isometric immersion of M into a Riemannian m manifold $R^m(c)$ of constant sectional curvature c,

$$\breve{A}_\pi \leq \frac{n^2}{4}\,|H|^2 + b(n-b)c, \tag{1.3}$$

where $\breve{A}_\pi = \sum\limits_{i=1}^{m} \sum\limits_{s=b+1}^{m} \|A_{X_i} V_s\|^2$, $\{X_1, \ldots, X_b\}$ is an orthonormal basis of horizontal space and $\{V_{b+1}, \ldots, V_m\}$ is an orthonormal basis of vertical space.

In [1], P. Alegre, B.-Y. Chen and M. I. Munteanu gave an optimal equality involving δ-invariants for submanifolds of the complex projective space $CP^m(4)$ via Hopf's fibration $\pi : S^{2m+1} \to CP^m(4)$ as follows:

$$\delta^H \leq \frac{n^2(n-2)}{2(n-1)}\,\|H\|^2 + \|P\|^2 + \frac{1}{2}\left(n^2 - n - 2\right) \tag{1.4}$$

where $\|H\|^2$ is the squared mean curvature of submanifold N in $CP^m(4)$.

Pursuing the development of this research program, B.-Y. Chen formulated a series of fundamental questions in [6,7]. The first problem we study is the following:

Problem 1 How can we establish simple relationship between the main intrinsic invariants and the main extrinsic invariants of the vertical spaces and horizontal space of a Riemannian manifold admitting a Riemannian submersion?

It is known that there exist some Euclidean manifolds which admit non trivial submersion. In [9], B.-Y. Chen showed that under which condition a Riemannian submersion is non trivial. In this paper, we investigate under which condition a Riemannian submersion is non-trivial with the aid of some inequalities involving curvature invariants of the base manifold and target manifold admitting a Riemannian submersion.

The second problem we study is the following:

Problem 2 Which conditions are necessary for a Riemannian manifold admits a harmonic Riemannian submersion?

In [20], S. W. Wei showed that all fibers of the target space of a Riemannian submersion are minimal if and only if the submersion is harmonic. Using this important result of S. W. Wei, we try to respond Problem 2 by establishing some inequalities involving the curvature invariants of the base manifold and of the target manifold of a Riemannian submersion.

The paper is organized as follows: Section 2 is concerned with preliminaries. In section 3, we establish sharp inequalities on a Riemannian manifold which admit a Riemannian submersions. By using these inequalities, we obtain necessary and sufficient conditions for horizontal distribution $\mathcal{H}$ to be integrable and the fibres of Riemannian submersions to be either totally geodesic or totally umbilical.

2 Preliminaries

Let (M, g) and (B, g') be C^∞-Riemannian manifolds of dimension m and n respectively. A surjective C^∞ map $\pi : M \to B$ is a C^∞*submersion* if it has maximal rank at any point of M. For any $x \in B$, $\pi^{-1}(x)$ is closed r-dimensional $(r = m - n)$ submanifold of M. Putting $\mathcal{V}_p = \ker \pi_{*p}$ for any $p \in M$, we obtain an integrable distribution $\mathcal{V}$ which corresponds to the foliation of M determined by the fibres of π, since each $\mathcal{V}_p$ coincides with the tangent space of $\pi^{-1}(x)$ at p, $\pi(x) = p$. Each $\mathcal{V}_p$ is called the vertical space at p, $\mathcal{V}$ is the *vertical distribution*, the sections of $\mathcal{V}$ are called *vertical vector fields* and determine a Lie subalgebra $\chi^v(M)$ of $\chi(M)$. Let $\mathcal{H}$ be the complementary distribution of $\mathcal{V}$ determined by the Riemannian metric g. Then one has following orthogonal decomposition

$$T_p(M) = \mathcal{V}_p \oplus \mathcal{H}_p. \tag{2.1}$$

Here, $\mathcal{H}_p$ is called the horizontal space at p. Furthermore, π is called a Riemannian submersion if π_{*p} preserves the length of the horizontal vectors at each point p of M [11].

Let h and v are the projection morphisms of $\chi(M)$ on $\chi^h(M)$ and $\chi^v(M)$, respectively. O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ are defined by, respectively,

$$\mathcal{A}_E F = h\nabla_{hE} vF + v\nabla_{hE} hF, \tag{2.2}$$

and

$$\mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF, \tag{2.3}$$

for any $E, F \in \chi(M)$. Here, ∇ is the Levi-Civita connection of g and both $\mathcal{T}_E$, $\mathcal{A}_E$ are skew-symmetric operators on $\chi(M)$ [17].

It is known that the tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy that

$$\mathcal{T}_U W = \mathcal{T}_W U, \tag{2.4}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X, \tag{2.5}$$

for any $U, W \in \chi^v(M)$ and $X, Y \in \chi^h(M)$. From (2.2) and (2.3) equalities, one has

$$\nabla_V W = \mathcal{T}_V W + \widehat{\nabla}_V W, \tag{2.6}$$

$$\nabla_V X = h\nabla_V X + \mathcal{T}_V X, \tag{2.7}$$

$$\nabla_X V = \mathcal{A}_X V + v\nabla_X V, \tag{2.8}$$

$$\nabla_X Y = h\nabla_X Y + \mathcal{A}_X Y, \tag{2.9}$$

for any $V, W \in \chi^v(M)$ and $X, Y \in \chi^h(M)$ [11].

We denote the Riemannian curvature tensors of (M, g) and (B, g') by $\mathcal{R}$ and $\mathcal{R}'$, respectively. $\mathcal{R}$ and $\mathcal{R}'$ are given by, respectively,

$$\pi_*(\mathcal{R}^*(X, Y, Z)) = \mathcal{R}'(\pi_* X, \pi_* Y, \pi_* Z)$$

for any $X, Y, Z \in \chi^h(M)$.

From the Gauss and Codazzi equations, it is also known that

$$\mathcal{R}(U, V, F, W) = \widehat{\mathcal{R}}(U, V, F, W) - g(\mathcal{T}_U W, \mathcal{T}_V F) + g(\mathcal{T}_V W, \mathcal{T}_U F) \tag{2.10}$$

$$\mathcal{R}(U, V, W, X) = -g((\nabla_V \mathcal{T})(U, W), X) + g((\nabla_U \mathcal{T})(V, W), X) \tag{2.11}$$

for any $U, V, W, F \in \chi^v(M)$ and $X \in \chi^h(M)$.

Moreover, there are some relations involving curvature tensor $\mathcal{R}$ as follows:

$$\begin{aligned} \mathcal{R}(X, Y, Z, H) = \mathcal{R}^*(X, Y, Z, H) &+ 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ &-g(\mathcal{A}_Y Z, \mathcal{A}_X H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H), \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \mathcal{R}(X, V, Y, W) = &-g((\nabla_X \mathcal{T})(V, W), Y) - g((\nabla_V \mathcal{A})(X, Y), W) \\ &+g(\mathcal{T}_V X, \mathcal{T}_W Y) - g(\mathcal{A}_X V, \mathcal{A}_Y W), \end{aligned} \tag{2.13}$$

for any $X, Y, Z, H \in \chi^h(M)$ and $V, W \in \chi^v(M)$ [8].

Let α be a vertical 2-plane spanned by orthonormal vectors U and V. From (2.10), one has the following equality between the sectional curvature $\mathcal{K}(\alpha)$ and sectional curvature $\widehat{\mathcal{K}}(\alpha)$ on the fibre through p:

$$\mathcal{K}(\alpha) = \widehat{\mathcal{K}}(\alpha) - \|\mathcal{T}_U V\|^2 + g(\mathcal{T}_U U, \mathcal{T}_V V). \tag{2.14}$$

Let $\{X, Y\}$ be an orthonormal basis of the horizontal 2-plane α, $\mathcal{K}^{'}(\alpha')$ denotes the sectional curvature in (B, g') of the plane α' spanned by $\{\pi_* X,\ \pi_* Y\}$. From (2.12) one can obtain

$$\mathcal{K}(\alpha) = \mathcal{K}'(\alpha') + 3\left\|\mathcal{A}_X Y\right\|^2. \tag{2.15}$$

Finally if $X \in \chi^h(M)$, $V \in \chi^v(M)$ are unit vectors spanning α, then from (2.13) one can write

$$\mathcal{K}(\alpha) = -g\left(\left(\nabla_X \mathcal{T}\right)(V, V), X\right) + \left\|\mathcal{T}_V X\right\|^2 - \left\|\mathcal{A}_X V\right\|^2. \tag{2.16}$$

Here $\{X_i, U_j\}$ $(1 \leq i \leq n,\ 1 \leq j \leq r)$ is called π-*adapted a local orthonormal frame* on M [18].

From the above equations, scalar curvature $\tau(p)$ at $p \in M$ is given by

$$\tau(p) = \sum_{1 \leq i < j \leq r} \mathcal{K}(U_i, U_j) + \sum_{1 \leq i < j \leq n} \mathcal{K}(X_i, X_j) + \sum_{j=1}^{r}\sum_{i=1}^{n} \mathcal{K}(U_j, X_i). \tag{2.17}$$

The mean curvature vector field $H(p)$ on any fibre is given by

$$\mathcal{N} = rH, \tag{2.18}$$

where

$$\mathcal{N} = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j. \tag{2.19}$$

Let $\{U_1, U_2, ..., U_r\}$ be an orthonormal basis of $\chi^v(M)$. It is known that

$$g(\nabla_E \mathcal{N}, X) = \sum_{j=1}^{r} g((\nabla_E \mathcal{T})(U_j, U_j), X), \tag{2.20}$$

for any $E \in \chi(M)$ and $X \in \chi^h(M)$ [11].

The horizontal divergence of any vector field X on $\chi^h(M)$ is shown by $\check{\delta}(X)$ and defined by

$$\check{\delta}(X) = \sum_{i=1}^{n} g(\nabla_{X_i} X, X_i), \tag{2.21}$$

where $\{X_1, X_2, ..., X_n\}$ is an orthonormal basis of $\chi^h(M)$. Thus, we put

$$\check{\delta}(\mathcal{N}) = \sum_{i}^{n} \sum_{i=1}^{r} g((\nabla_{X_i} \mathcal{T})(U_j, U_j), X_i. \tag{2.22}$$

For details, we refer to [2].

It is said to be π *has totally geodesic fibres* if $\mathcal{T}$ vanishes identically and π has totally umbilical fibres if

$$\mathcal{T}_U V = g(U, V) H, \tag{2.23}$$

where $U, V \in \chi^v(M)$ and H is mean curvature vector field of fibres.

A map π is called $p-$ *harmonic*, $p \geq 1$, if it is a weak solution to the following Euler-Lagrange equation for E_p :

$$\begin{aligned} r_p(\pi) &:= div\left(|d\pi|^{p-2} d\pi\right) := \sum_{i=1}^{n} \left(\nabla^{\pi}_{e_i} |d\pi|^{p-2} d\pi\right)(e_i) \\ &= \sum_{i=1}^{n} \left(\nabla^{\pi}_{e_i} |d\pi|^{p-2} d\pi(e_i) - |d\pi|^{p-2} d\pi\left(\nabla^{M}_{e_i} e_i\right)\right) \\ &= 0, \end{aligned}$$

where $r_p(\pi)$ is the $p-$tension field of π, ∇^{π} is the pullback connection of the Levi-Civita connection ∇^B on B, on the induced bundle $\pi^{-1}(TB)$. That is, a C^1 map $\pi : M \to B$ satisfies

$$\sum_{i=1}^{n} \int_M g'(|d\pi|^{p-2} d\pi(e_i), \nabla^{\pi}_{e_i} V) dv = 0,$$

for every smooth compactly supported vector field V along the map π. For $p = 2$, π is called *harmonic morphism*. For more details, we refer to [15] and [20].

3 Some Inequalities for Riemannian Submersions

In this section, we study the scalar curvature of a Riemannian manifold admitting a Riemannian submersion and we establish some inequalities involving the scalar curvatures of (M^m, g) and (B^n, g').

We begin this section with the following lemma:

Lemma 3.1 *Let (M^m, g) and (B^n, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. Then we have*

$$\begin{aligned} 2\tau(p) = 2\widehat{\tau}(p) + 2\tau^*(p) - \|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|^2 + r^2\|H\|^2 + 3\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|^2 \\ -\check{\delta}(\mathcal{N}) + \|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\|^2 - \|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\|^2, \end{aligned} \tag{3.1}$$

where

$$\widehat{\tau}(p) = \sum_{1\leq i<j\leq r} \widehat{\mathcal{K}}(U_i, U_j)$$

is the scalar curvature of the vertical space of M and

$$\tau^*(p) = \sum_{1\leq i<j\leq n} \mathcal{K}^*(X_i, X_j)$$

is the scalar curvature of the horizontal space of M.

Proof. If we put (2.10), (2.12) and (2.13) equalities in

$$\begin{aligned} \tau(p) = \tfrac{1}{2}\sum_{i,j=1}^{r} \mathcal{R}(U_i, U_j, U_j, U_i) + \tfrac{1}{2}\sum_{i,j=1}^{n} \mathcal{R}(X_i, X_j, X_j, X_i) \\ + \tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{r} \mathcal{R}(X_i, U_j, U_j, X_i), \end{aligned} \tag{3.2}$$

we have

$$
\begin{aligned}
2\tau(p) = & \sum_{i,j=1}^{r} \widehat{\mathcal{R}}(U_i, U_j, U_j, U_i) - \sum_{i,j=1}^{r}\sum_{s=1}^{n}\left(\mathcal{T}_{ij}^{s}\right)^2 + \sum_{i,j=1}^{r}\sum_{s=1}^{n}\mathcal{T}_{ii}^{s}\mathcal{T}_{jj}^{s} \\
+ & \sum_{i,j=1}^{n} \mathcal{R}^*(X_i, X_j, X_j, X_i) + 3\sum_{i,j=1}^{n}\sum_{s=1}^{r}\left(\mathcal{A}_{ij}^{s}\right)^2 \\
- & \sum_{j=1}^{r}\sum_{i=1}^{n} g\left(\left(\nabla_{X_i}\mathcal{T}\right)(U_j, U_j), X_i\right) + \sum_{j=1}^{r}\sum_{i=1}^{n}\sum_{s=1}^{r}\left(\mathcal{T}_{ji}^{s}\right)^2 \\
- & \sum_{j=1}^{r}\sum_{i=1}^{n}\sum_{s=1}^{r}\left(\mathcal{A}_{ij}^{s}\right)^2 .
\end{aligned}
\tag{3.3}
$$

From (2.22), we get (3.1). □

Using Lemma 3.1, we get the following theorems and corollary immediately:

Theorem 3.2 *Let (M^m, g) and (B^n, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. Then we have*

$$
\begin{aligned}
2\tau(p) \geq & \, 2\widehat{\tau}(p) + 2\tau^*(p) - \|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|^2 + r^2\|H\|^2 \\
& - \check{\delta}(\mathcal{N}) - \|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\|^2 + \|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\|^2
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
2\tau(p) \leq & \, 2\widehat{\tau}(p) + 2\tau^*(p) - \|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|^2 + r^2\|H\|^2 \\
& - \check{\delta}(\mathcal{N}) + \|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\|^2 + 3\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|^2
\end{aligned}
\tag{3.5}
$$

Equality cases of (3.4) *and* (3.5) *hold for all $p \in M$ if and only if horizontal distribution $\mathcal{H}$ is integrable.*

From Theorem 3.2, we have the following corollary immediately:

Corollary 3.3 *Let $\pi : (M^m, g) \to (B^n, g')$ be a Riemannian submersion with totally geodesic fibres. Then, we have*

$$
2\tau(p) \geq 2\widehat{\tau}(p) + 2\tau^*(p) - \|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\|^2 . \tag{3.6}
$$

and

$$
2\tau(p) \leq 2\widehat{\tau}(p) + 2\tau^*(p) + 3\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|^2 . \tag{3.7}
$$

Equality cases of (3.6) *and* (3.7) *hold for all $p \in M$ if and only if horizontal distribution $\mathcal{H}$ is integrable.*

Theorem 3.4 *Let (M^m, g) and (B^n, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. Then we have*

$$
\begin{aligned}
2\tau(p) \leq & \, 2\widehat{\tau}(p) + 2\tau^*(p) + r^2\|H\|^2 + 3\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|^2 \\
& - \check{\delta}(\mathcal{N}) - \|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\|^2 + \|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\|^2 .
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
2\tau(p) \geq & \, 2\widehat{\tau}(p) + 2\tau^*(p) + r^2\|H\|^2 + 3\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|^2 \\
& - \check{\delta}(\mathcal{N}) - \|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\|^2 - \|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|^2 .
\end{aligned}
\tag{3.9}
$$

Equality cases of (3.8) *and* (3.9) *hold for all $p \in M$ if and only if the fibre through p of π is a totally geodesic submanifold of M.*

From Theorem 3.4, we have the following corollary immediately:

Corollary 3.5 *Let $\pi : (M^m, g) \to (B^n, g')$ be a Riemannian submersion and horizontal distribution $\mathcal{H}$ is integrable. Then, we have*

$$2\tau(p) \leq 2\widehat{\tau}(p) + 2\tau^*(p) + \|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\|^2 + r^2\|H\|^2 - \check{\delta}(\mathcal{N}). \tag{3.10}$$

and

$$2\tau(p) \geq 2\widehat{\tau}(p) + 2\tau^*(p) - \|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|^2 + r^2\|H\|^2 - \check{\delta}(\mathcal{N}). \tag{3.11}$$

Equality cases of (3.10) *and* (3.11) *hold for all $p \in M$ if and only if the fibre through p of π is a totally geodesic submanifold of M.*

Using by the arithmetic mean-geometric mean (AM-GM) inequality, we get the following two theorems:

Theorem 3.6 *Let (M^m, g) and (B^n, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. Then we have*

$$\begin{aligned} 2\tau(p) \leq\ & 2\widehat{\tau}(p) + 2\tau^*(p) - 2\|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|\,\|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\| \\ & + r^2\|H\|^2 + 3\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|^2 - \check{\delta}(\mathcal{N}) + \|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\|^2. \end{aligned} \tag{3.12}$$

Equality case of (3.12) *holds for all $p \in M$ if and only if $\|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\| = \|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|$ or both of dimensions of the fibre of π and horizontal distribution are the same.*

Theorem 3.7 *Let (M^m, g) and (B^n, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. Then we have*

$$\begin{aligned} 2\tau(p) \geq\ & 2\widehat{\tau}(p) + 2\tau^*(p) + 2\sqrt{3}\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|\,\|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\| \\ & - \|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|^2 + r^2\|H\|^2 - \check{\delta}(\mathcal{N}) - \|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\|^2. \end{aligned} \tag{3.13}$$

Equality case of (3.13) *holds for all $p \in M$ if and only if $\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\| = \|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|$ or both of dimensions of the fibre of π and horizontal distribution are the same.*

Using by Cauchy-Schwarz inequality (see also Lemma 3.2 in [19]), we have the following theorem:

Theorem 3.8 *Let (M^m, g) and (B^n, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. Then we have*

$$\begin{aligned} & 2\tau(p) \leq 2\widehat{\tau}(p) + 2\tau^*(p) - r(1-r)\|H\|^2 + 3\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|^2 \\ & -\check{\delta}(\mathcal{N}) - \|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\|^2 + \|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\|^2. \end{aligned} \tag{3.14}$$

Equality case of (3.14) *holds for all $p \in M$ if and only if we have the following statements:*
i) π *is a Riemannian submersion that has a totally umbilical fibres.*
ii) $\mathcal{T}_{ij} = 0$, *for* $i \neq j \in \{1, 2, \ldots, r\}$.

Proof. From (3.1) we have

$$\begin{aligned} 2\tau(p) &= 2\widehat{\tau}(p) + 2\tau^*(p) - \sum_{s=1}^{n}\sum_{i=1}^{r}(\mathcal{T}_{ii}^s)^2 - \sum_{s=1}^{n}\sum_{i\neq j}^{r}\left(\mathcal{T}_{ij}^s\right)^2 \\ &+\sum_{s=1}^{n}\sum_{i,j=1}^{r}\mathcal{T}_{ii}^s\mathcal{T}_{jj}^s + 3\left\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\right\|^2 - \check{\delta}(\mathcal{N}) \\ &+\left\|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\right\|^2 - \left\|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\right\|^2. \end{aligned} \tag{3.15}$$

From Cauchy-Schwarz inequality, we get

$$\begin{aligned} 2\tau(p) &\leq 2\widehat{\tau}(p) + 2\tau^*(p) - \tfrac{1}{r}\sum_{s=1}^{n}\left(\sum_{i=1}^{r}\mathcal{T}_{ii}^r\right)^2 - \sum_{s=1}^{n}\sum_{i\neq j}^{r}\left(\mathcal{T}_{ij}^s\right)^2 \\ &+ r^2\left\|H\right\|^2 + 3\left\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\right\|^2 - \check{\delta}(\mathcal{N}) \\ &+\left\|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\right\|^2 - \left\|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\right\|^2, \end{aligned} \tag{3.16}$$

which is equivalent to (3.14). Equality case of (29) holds for all $p \in M$ if and only if

$$\mathcal{T}_{11} = \mathcal{T}_{22} = ... = \mathcal{T}_{rr} \quad \text{and} \quad \sum_{s=1}^{n}\sum_{i\neq j}^{r}\left(\mathcal{T}_{ij}^s\right)^2 = 0, \tag{3.17}$$

which completes proof of the theorem □

Using by similar proof way of Theorem 3.8, we have the following theorem:

Theorem 3.9 *Let (M^m, g) and (B^n, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$. Then we have*

$$\begin{aligned} 2\tau(p) \geq{}& 2\widehat{\tau}(p) + 2\tau^*(p) - \left\|\mathcal{T}_{V\times V}\right\|^2 + r^2\left\|H\right\|^2 \\ &+\tfrac{3}{n}\mathrm{trace}\left(\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\right)^2 - \check{\delta}(\mathcal{N}) + \left\|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\right\|^2 - \left\|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\right\|^2. \end{aligned} \tag{3.18}$$

Equality case of (3.18) *holds for all $p \in M$ if and only if $\mathcal{A}_{11} = \mathcal{A}_{22} = \ldots = \mathcal{A}_{nn}$ and $\mathcal{A}_{ij} = 0$, for $i \neq j \in \{1, 2, ..., n\}$.*

Corollary 3.10 *Let $\pi : (M^m, g) \to (B^n, g')$ be a Riemannian submersion with totally geodesic fibres. Then, we have*

$$2\tau(p) \geq 2\widehat{\tau}(p) + 2\tau^*(p) + \tfrac{3}{n}\mathrm{trace}\left(\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\right)^2. \tag{3.19}$$

Equality case of (3.19) *holds for all $p \in M$ if and only if $\mathcal{A}_{11} = \mathcal{A}_{22} = \ldots = \mathcal{A}_{nn}$ and $\mathcal{A}_{ij} = 0$, for $i \neq j \in \{1, 2, ..., n\}$.*

Theorem 3.11 [9] *Let $\pi : M \to B$ be a Riemannian submersion. Then π is $p-$harmonic, for every $p > 1$, if and only if all fibers $\pi^{-1}(x)$, $x \in B$ are minimal submanifolds in M.*

Using Theorem 3.11, we have the following theorem.

Theorem 3.12 *Equality cases of inequalities* (3.8) *and* (3.9), (3.10) *and* (3.11), (3.14) *are hold for all $p \in M$ if and only if Riemannian submersion π is harmonic.*

Now, we shall need the following lemma [4]:

Lemma 3.13 *If $n > k \geq 2$ and $a_1, \ldots, a_n, a$ are real numbers such that*

$$(\sum_{i=1}^{n} a_i)^2 = (n-k+1)(\sum_{i=1}^{n} (a_i)^2 + a), \tag{3.20}$$

then

$$2 \sum_{1\leq i<j\leq k} a_i a_j \geq a,$$

with equality holding if and only if

$$a_1 + a_2 = ... = a_k = a_{k+1} = \cdots = a_n.$$

Theorem 3.14 *Let (M^m, g) and (B^n, g') be Riemannian manifolds admitting a Riemannian submersion $\pi : M \to B$ and denote by $\mathcal{P}$ the plane spanned by U_1 and U_2 the vertical vectors. Then, one has:*

$$\begin{aligned} \tau(p) - \mathcal{K}_{\mathcal{P}} \leq \widehat{\tau}(p) + \tau^*(p) + \widehat{\mathcal{K}}_{\mathcal{P}} - \tfrac{r^2(r-2)}{2(r-1)} \|H\|^2 + \tfrac{1}{2}\check{\delta}(\mathcal{N}) \\ + \tfrac{1}{2}\|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\|^2 - \tfrac{1}{2}\|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\|^2 + \tfrac{3}{2}\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|^2 . \end{aligned} \tag{3.21}$$

Equality case of (3.21) *holds for all $p \in M$ if and only if there exists an orthonormal basis $\{U_1, U_2, ..., U_r\}$ of $\chi^v(M)$ and an orthonormal basis $\{X_1, X_2, ..., X_n\}$ of $\chi^h(M)$ such that shape operators $S_{X_1}, \ldots, S_{X_n}$ of the vertical space of M take the following forms:*

$$S_{X_1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda \end{pmatrix}, \quad \lambda = a + b$$

and

$$S_{X_s} = \begin{pmatrix} c_s & d_s & \ldots & 0 \\ d_s & -c_s & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \quad 2 \leq s \leq n.$$

Proof. If we put

$$\begin{aligned} \omega = 2\widehat{\tau}(p) + 2\tau^*(p) - 2\tau(p) + \tfrac{r^2(r-2)}{r-1}\|H\|^2 + \|\mathcal{T}_{\mathcal{V}\times\mathcal{H}}\|^2 \\ - \|\mathcal{A}_{\mathcal{H}\times\mathcal{V}}\|^2 + 3\|\mathcal{A}_{\mathcal{H}\times\mathcal{H}}\|^2 - \check{\delta}(\mathcal{N}). \end{aligned} \tag{3.22}$$

in (3.1) we have

$$\omega = \|\mathcal{T}_{\mathcal{V}\times\mathcal{V}}\|^2 - \tfrac{r^2}{r-1}\|H\|^2 . \tag{3.23}$$

Moreover, we choose the normal vector X_1 to be in the direction of the mean curvature vector at p.and from (3.23) we can write

$$\left(\sum_{i=1}^{r}\mathcal{T}_{ii}^{1}X_1\right)^2 = (r-1)(-w+\sum_{i=1}^{r}(\mathcal{T}_{ii}^{1})^2+\sum_{i\neq j=1}^{r}(\mathcal{T}_{ij}^{1})^2 + \sum_{s=2}^{n}\sum_{i,j=1}^{r}(\mathcal{T}_{ij}^{s})^2). \tag{3.24}$$

Applying Lemma 3.13, we get

$$\mathcal{T}_{11}^{1}\mathcal{T}_{22}^{1} \geq -\frac{\omega}{2}+\frac{1}{2}\sum_{i\neq j=1}^{r}\left(\mathcal{T}_{ij}^{1}\right)^2+\frac{1}{2}\sum_{s=2}^{n}\sum_{i,j=1}^{r}\left(\mathcal{T}_{ij}^{s}\right)^2. \tag{3.25}$$

Let we choose a 2-plane section π spanned by orthonormal vectors U_1 and U_2 on $\chi^v(M)$. From (2.10) and (3.25), we obtain the followings:

$$\begin{aligned}\mathcal{K}(\alpha) &\leq \widehat{\mathcal{K}}(\alpha)+\sum_{s=1}^{n}(\mathcal{T}_{12}^{s})^2-\frac{\omega}{2}+\frac{1}{2}\sum_{i\neq j=1}^{r}(\mathcal{T}_{ij}^{1})^2 \\ &\quad +\frac{1}{2}\sum_{s=2}^{n}\sum_{i,j=1}^{r}(\mathcal{T}_{ij}^{s})^2+\sum_{s=2}^{n}\mathcal{T}_{11}^{s}\mathcal{T}_{22}^{s} \\ &\leq \widehat{\mathcal{K}}(\alpha)-\frac{\omega}{2}+\frac{1}{2}\sum_{s=2}^{n}(\mathcal{T}_{11}^{s}+\mathcal{T}_{22}^{s})^2 \\ &\quad +\frac{1}{2}\sum_{s=2}^{n}\sum_{i,j>2}(\mathcal{T}_{ij}^{s})^2+\sum_{s=1}^{n}\sum_{j>2}^{r}(\left(\mathcal{T}_{1j}^{s}\right)^2+\left(\mathcal{T}_{2j}^{s}\right)^2) \\ &\leq \widehat{\mathcal{K}}(\alpha)-\frac{\omega}{2}\end{aligned} \tag{3.26}$$

which is equivalent to (3.21). Equality case of (3.21) holds for all $p \in M$ if and only if

$$\sum_{s=2}^{n}(\mathcal{T}_{11}^{s}+\mathcal{T}_{22}^{s})^2=0, \quad \sum_{s=1}^{n}\sum_{j>1}^{r}\left(\left(\mathcal{T}_{1j}^{s}\right)^2+\left(\mathcal{T}_{2j}^{s}\right)^2\right)=0, \quad \sum_{s=2}^{n}\sum_{i,j>2}^{r}\left(\mathcal{T}_{ij}^{s}\right)^2=0$$

and

$$\mathcal{T}_{11}^{1}+\mathcal{T}_{22}^{1}=\mathcal{T}_{33}^{1}=...=\mathcal{T}_{rr}^{1}$$

which follows that shape operators $S_{X_1},\dots,S_{X_n}$ of the vertical space of M at p take the desired form. □

Example 3.1 Let $\pi:\mathbb{S}^{2n+1}\to\mathbb{CP}^n(4)$ be a Riemannian submersion, where $\mathbb{S}^{2n+1}$ has constant curvature 1 and complex projective n-space $\mathbb{CP}^n(4)$ of constant holomorphic sectional curvature 4. It's clear that complex Hopf fibration π satisfy the inequality (3.21).

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References

1. Alegre, P.; Chen B.-Y.; Munteanu, M.I. – *Riemannian submersions, δ-invariants, and optimal inequality*, Ann. Glob. Anal. Geom., **42** (2012), no. 3, 317–331.
2. Besse, A.L. – *Einstein Manifolds.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 10. Springer-Verlag, Berlin, 1987. xii+510 pp.
3. Cartan, E. – *Sur la possibilité de plonger un espace Riemannian donné dans un espace Euclidien*, Ann. Soc. Pol. Math., **6** (1927), 1–7.
4. Chen, B.-Y. – *Some pinching and classification theorems for minimal submanifolds*, Arch. Math.(Basel), **60** (1993), no. 6, 568–578.
5. Chen, B.-Y. – *Mean curvature and shape operator of isometric immersions in real-space-forms*, Glasgow Math. J., **38** (1996), no. 1, 87–97.
6. Chen, B.-Y. – *Riemannian DNA, Inequalities and Their Applications*, Tamkang Journal of Science and Engineering, Vol. **3**, (2000) no. 3, pp. 123-130.
7. Chen, B.-Y. – *What can we do with Nash's embedding theorem?*, Soochow J. Math., **30** (2004), no. 3, 303–338.
8. Chen, B.-Y. – *Pseudo-Riemannian geometry, δ-invariants and applications. With a foreword by Leopold Verstraelen*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2011), xxxii+477 pp.
9. Chen, B.-Y. – *Riemannian submersions, minimal immersions and cohomology class*, Proc. Japan Acad. Ser. A Math. Sci., **81** (2005), no. 10, 162–167.
10. Chen, B.-Y.; Okumura, M. – *Scalar curvature, inequality and submanifold*, Proc. Amer. Math. Soc., **38** (1973), 605–608.
11. Falciteli, M.; Ianus, S.; Pastore, A.M. – *Riemannian Submersions and Related Topics*, World Scientific Publishing Co. Pte. Ltd., Inc., River Edge, NJ, (2004), xiv+277 pp.
12. Gauss, C.F. – *Disquisitiones generales circa superficies curvas*, Comment. Soc. Sci. Gotting. Recent. Classis Math. **6** (1828).
13. Gray, A. – *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech., **16** (1967), 715–737.
14. Janet, M. – *Sur la possibilité de plonger un espace Riemannian donné dans un espace Euclidien*, Ann. Soc. Polon. Math., **5** (1926), 38-43.
15. Loubeau, E. – *On p-harmonic morphisms*, Differential Geom. Appl., **12** (2000), no. 3, 219–229.
16. Nash, J.N. – *The imbedding problem for Riemannian manifolds*, Annals of Math. (2), **63** (1956), 20–63.
17. O'Neill, B. – *The fundamental equations of a submersion*, Michigan Math. J., **13** (1966), 459–469.
18. Şahin, B. – *Manifoldların Diferensiyel Geometrisi*, Nobel Akademik Yayıncılık Eğitim Danışmanlık tic. ltd. şti., 2012.
19. Tripathi, M.M. – *Certain basic inequalities for submmanifolds in (κ, μ)-spaces*, Recent advances in Riemannian and Lorentzian geometries, (Baltimore, MD, 2003), Editors K. L. Duggal and R. Sharma, Amer. Math. Soc., Providence, RI, Contemp. Math., **337** (2003), 187–202.
20. Wei, S.W. – *The unity of p-harmonic geometry*, Recent developments in geometry and analysis, Adv. Lect. Math. (ALM), 23, Int. Press, Somerville, MA, (2012), 439–483.