Mediterr. J. Math. 13 (2016), 2185–2198
DOI 10.1007/s00009-015-0613-4
1660-5446/16/042185-14
published online July 25, 2015

Mediterranean Journal of Mathematics

Conformality on Semi-Riemannian Manifolds

Cornelia-Livia Bejan and Şemsi Eken

Dedicated to Professor Anna-Maria Pastore on the occasion of her 70th birthday

Abstract. We introduce here the notion of conformal semi-Riemannian map between semi-Riemannian manifolds aiming to unify and generalize two geometric concepts. The first one is studied by García-Río and Kupeli (namely, semi-Riemannian map between semi-Riemannian manifolds). The second notion is defined by Şahin (namely, conformal Riemannian map between Riemannian manifolds) as an extension of the notion of Riemannian map introduced by Fischer. We support the main notion of this paper with several classes of examples, e.g. semi-Riemanninan submersions (see O'Neill's book and Falcitelli, Ianus and Pastore's book) and isometric immersions between semi-Riemannian manifolds. As a tool, we use the screen distributions (specific in semi-Riemannian geometry) of Duggal and Bejancu's book to obtain some characterizations and to give a semi-Riemannian version of Fischer's (resp. Şahin's) results, using the new map introduced here. We study the generalized eikonal equation and at the end relate the main notion of the paper with harmonicity.

Mathematics Subject Classification. 53C20, 53C43, 53C50, 58E20.

Keywords. Riemannian submersion, Semi-Riemannian map, Conformal semi-Riemannian map, Harmonic map.

1. Introduction

In the Riemannian context, two fundamental notions, namely the Riemannian maps introduced by Fischer [7] on one side and horizontally conformal maps given by Fuglede [8] and Ishihara [13] on the other side, were both generalized by conformal Riemannian maps defined in [19]. Şahin motivated there the importance of this new class of maps between Riemannian manifolds by several geometric properties and practical applications in computer vision, computer graphics and medical imaging fields (see [12,16,20]). Conformal maps between cortical surfaces were computed in [21].

Birkhäuser

Next, Fuglede extended the notion of horizontally conformal map from the Riemannian context (see [8]) to the semi-Riemannian one (see [9]) with the purpose of characterizing harmonic morphisms between semi-Riemannian manifolds (see [2]). Some theoretical applications to gravity of these horizontally conformal maps between semi-Riemannian manifolds were provided by Mustafa in [17]. Moreover, these maps were described in terms of jets in [14]. The class of horizontally conformal maps contains in particular semi-Riemannian submersions, for which we refer to [6] and [18]. Semi-Riemannian submersions are generalized by the semi-Riemannian maps between semi-Riemannian manifolds. The importance of this subject in semi-Riemannian geometry was exposed by García-Río and Kupeli in their monograph [10], devoted to the study of the semi-Riemannian maps between semi-Riemannian manifolds.

Our goal is to introduce in this paper a new class of maps between semi-Riemannian manifolds with the purpose of unifying and generalizing the above two concepts, namely the one treated in [19] (i.e. conformal Riemannian maps between Riemannian manifolds) and the other one studied in [10] (i.e. semi-Riemannian maps between semi-Riemannian manifolds). This class of maps, which we call conformal semi-Riemannian maps between semi-Riemannian manifolds contains semi-Riemannian submersions (see [6] and [11]) and isometric immersions between semi-Riemannian manifolds as particular cases. Different from the approach of [10] by using quotient spaces, in our approach we use the screen distributions introduced by [5], which we present in Sect. 2. Next, we characterize the semi-Riemannian maps between semi-Riemannian manifolds and show some properties of them in Sect. 3. The main notion of our paper, namely conformal semi-Riemannian map between semi-Riemannian manifolds, is given by Sect. 4 which provides several classes of examples. Section 5 is devoted to the generalized eikonal equation. As it was mentioned in ([10], page 92), Fischer's result for Riemannian map (and similar for Şahin's result on conformal Riemannian map) is not valid in the semi-Riemannian case. By using conformal semi-Riemannian maps defined here, we adapt both these results so that they remain valid in the semi-Riemannian context. The last section relates this new notion of conformality with that of harmonicity used in many branches of mathematics.

We assume throughout this paper the manifolds and maps to be smooth.

2. Preliminaries

We recall now some notions of [5]:

Notations 2.1. Let $F : (M^m, g) \to (N^n, h)$ be a map between semi-Riemannian manifolds. At any point $p \in M$, one has the following linear spaces:

$$\begin{aligned} V_p &= \{X \in T_pM \mid F_{*p}X = 0\} = kerF_{*p}, \\ H_p &= \{Y \in T_pM \mid g(Y, X) = 0\} = V_p^{\perp}, \\ Rad(V_p) &= V_p \cap H_p, \end{aligned}$$

which denote respectively the vertical, the horizontal and the radical space. In the Riemannian case, we don't need the following assumption, but in the semi-Riemannian case, we make the assumption that we obtain a distribution which we call vertical (resp. horizontal) if we assign to each $p \in M \to V_p$ the vertical (resp. $p \in M \to H_p$ the horizontal) space:

$$V = \bigcup_{p\in M} V_p = kerF_*,$$
$$H = \bigcup_{p\in M} H_p = V^{\perp}.$$

Suppose that the mapping $p \in M \to Rad(V_p)$, which assigns to each $p \in M$ the radical subspace $Rad(V_p)$ of V_p with respect to g_p, defines a smooth distribution $Rad(V)$ of rank $r \in \mathbb{N}$ on M. Obviously, $Rad(V)$ is a totally degenerate distribution on M, since g restricted to $Rad(V)$ is identically zero.

We note that the leaves of the vertical distribution are lightlike (resp. semi-Riemannian) submanifolds of M provided $r > 0$ (resp. $r = 0$).

Consider a complementary distribution $S(V)$ to $Rad(V)$ in V. The fibres of $S(V)$ are $S(V_p)$ defined such that

$$V_p = Rad(V_p) \oplus S(V_p),$$

for any $p \in M$. As these fibres of $S(V)$ are screen subspaces of V_p, $p \in M$ (see [5]), we call $S(V)$ the vertical screen distribution on M.

Similarly, let $S(H)$ be a complementary distribution to $Rad(V)$ in H. The fibres of $S(H)$ are $S(H_p)$ defined such that

$$H_p = Rad(V_p) \oplus S(H_p),$$

for any $p \in M$. Analogously, we call $S(H)$ the horizontal screen distribution on M.

Let $\pi_H : H \to S(H)$ denote the projection of $H = Rad(V) \oplus S(H)$ on $S(H)$.

Claim. From now on, we assume that all screen distributions related to F are arbitrarily fixed.

Lemma 2.2. *The following properties hold good:*

(i) *The distribution $Rad(V)$ is degenerate, while $S(V)$ and $S(H)$ are nondegenerate.*
(ii) *We have $S(H) \perp V$ and $S(V) \perp H$, since $S(H_p) \perp V_p$ and $S(V_p) \perp H_p$, for any $p \in M$.*
(iii) $\dim V + \dim H = \dim M$.
(iv) $(V^{\perp})^{\perp} = H^{\perp} = V$.
(v) *The following equivalences hold: $(V, g_{/V})$ is a nondegenerate distribution $\Leftrightarrow Rad(V) = \{0\} \Leftrightarrow TM = V \oplus H$.*
(vi) *Any leaf of the vertical distribution V is either a lightlike submanifold of M (provided $(V, g_{/V})$ is degenerate) or a semi-Riemannian manifold (provided $(V, g_{/V})$ is nondegenerate).*

From the above Lemma 2.2 and from ([10], Proposition 1.1.2 and 1.1.3 page 5), we obtain the following:

Proposition 2.3. *If the vertical distribution* $(V, g_{/V})$ *is lightlike of type* (r, ν', η'), *then* H *is a lightlike distribution on* (M, g) *in* TM, *of type* $(r, \nu - r - \nu', m - \nu - r - \eta')$, *where* $m = \dim M$ *and* ν *is the index of* M. *Moreover,*

$$[Rad(V)]^{\perp}_{g_{/V}} = V + H$$

is a lightlike distribution on (M, g) *in* TM *of type* $(r, \nu - r, m - \nu - r)$.

Corollary 2.4. *In particular, when the vertical leaves are degenerate (i.e. lightlike) hypersurfaces of* (M, g) *(see* [1]*), then* $Rad(V) = H$. *Hence the horizontal distribution is of dimension 1 and* $(V, g_{/V})$ *is of type* $(1, \nu - 1, m - \nu - 1)$.

Notations 2.5. *Suppose that the mapping*

$$p \in M \to Rad(ImF_{*p}) = ImF_{*p} \cap (ImF_{*p})^{\perp}$$

which assigns to each $p \in M$ the radical subspace $Rad(ImF_{*p})$ of ImF_{*p} (with respect to h) is a vector bundle on M. Consider a complementary vector subbundle $S(ImF_*)$ to $Rad(ImF_*)$ (with respect to h) in

$$ImF_* = \bigcup_{p \in M} ImF_{*p}.$$

The fibres of $S(ImF_*)$ are $S(ImF_{*p})$ defined such that

$$ImF_{*p} = S(ImF_{*p}) \oplus Rad(ImF_{*p})$$

for any $p \in M$. We call $S(ImF_*)$ the screen vector subbundle of the image of F_* and let

$$\pi_{ImF_*} : ImF_* \to S(ImF_*)$$

denote the projection of $ImF_* = S(ImF_*) \oplus Rad(ImF_*)$ to the first component of the direct sum.

3. Semi-Riemannian Maps

Different from García-Río and Kupeli's book [10], which uses quotient spaces, in our approach we use complementary screen distributions and vector bundles to give the following:

Definition 3.1. Let $F : (M^m, g) \to (N^n, h)$ be a map between semi-Riemannian manifolds. For any $p \in M$, we introduce the screen tangent map F^S_{*p} defined as the restriction of F_{*p}:

$$F^S_{*p} = F_{*p/S(H_p)} : (S(H_p), g_{/S(H_p)}) \to (S(ImF_{*p}), h_{/S(ImF_{*p})}),$$

given by

$$F^S_{*p}(\overline{X}) = \pi_{ImF_{*p}}(F_{*p}X),$$

where $X \in H$ and $\pi_H(X) = \overline{X}$. The rank of the F^S_{*p} is called the nondegenerate rank of F in p.

Remark 3.2. (i) We note that at $p \in M$, the screen tangent map F^S_{*p} may be neither injective nor surjective.

(ii) For any $p \in M$, the linear transformation F^S_{*p} depends on the screen distribution, while the rank of F^S_{*p} is independent of it. Therefore, the nondegenerate rank of F at p is well defined.

Notations 3.3. Let $F : (M^m, g) \to (N^n, h)$ be a map between semi-Riemannian manifolds. Then the square norm of F is defined at any point $p \in M$ by

$$\|F_{*p}\|^2 =< F_{*p}, F_{*p} >= trace_g(F_{*p}\cdot, F_{*p}\cdot) = \sum_{i=1}^{m} \varepsilon_i h(F_{*p}u_i, F_{*p}u_i)$$

where $\{u_1, u_2, \ldots, u_m\}$ is an orthonormal basis in T_pM and $\varepsilon_i = g(u_i, u_i) \in \{-1, 1\},\ i = 1, \ldots, m$.

Lemma 3.4. *Let $F : (M^m, g) \to (N^n, h)$ be a map between semi-Riemannian manifolds and $p \in M$. Then,*

$$\|F_{*p}\|^2 = \|F^S_{*p}\|^2.$$

Proof. Let $\{f_1, \ldots, f_s\}$ be a local orthonormal frame of the screen vertical distribution $S(V)$, and $\{e_1, \ldots, e_t\}$ be a local orthonormal frame of the screen horizontal distribution $S(H)$. Note that $span\{f_1, \ldots, f_s, e_1, \ldots, e_t\}$ is a non-degenerate subspace in T_pM and denote by $(span\{f_1, \ldots, f_s, e_1, \ldots, e_t\})^\perp$ its orthogonal complemantary space in T_pM. Since g is a nondegenerate metric on M, it follows that $(span\{f_1, \ldots, f_s, e_1, \ldots, e_t\})^\perp$ is also a nondegenerate subspace and we may take $\{z_1, w_1, \ldots, z_k, w_k\}$ to be an orthonormal basis of it, such that

$$g(z_i, z_j) = \delta_{ij} = -g(w_i, w_j)$$

and

$$g(z_i, w_j) = 0, \quad \text{for any } i, j \in \{1, 2, \ldots, k\}.$$

So, $z_i + w_i \in Rad(V)$ for $i = 1, 2, \ldots, k$. Then, $\{f_1, \ldots, f_s, e_1, \ldots, e_t, z_1, w_1, \ldots, z_k, w_k\}$ is a local orthonormal frame on (M, g). Hence,

$$\begin{aligned}
\|F_{*p}\|^2 &= \sum_{i=1}^{s} g(f_i, f_i) g((^*F_{*p} \circ F_{*p}) f_i, f_i) + \sum_{i=1}^{t} g(e_i, e_i) g((^*F_{*p} \circ F_{*p}) e_i, e_i) \\
&\quad + \sum_{i=1}^{k} g(z_i, z_i) g((^*F_{*p} \circ F_{*p}) z_i, z_i) \\
&\quad + \sum_{i=1}^{k} g(w_i, w_i) g((^*F_{*p} \circ F_{*p}) w_i, w_i) \\
&= \sum_{i=1}^{t} g(e_i, e_i) h(F_{*p} e_i, F_{*p} e_i) + \sum_{i=1}^{k} g(z_i, z_i) h(F_{*p} z_i, F_{*p} z_i) \\
&\quad + \sum_{i=1}^{k} g(w_i, w_i) h(F_{*p} w_i, F_{*p} w_i).
\end{aligned}$$

But since for $i = 1, 2, \ldots, k$ we have $z_i + w_i \in Rad(V) \subseteq V$, then

$$0 = F_{*p}z_i + F_{*p}w_i$$

and

$$0 = g(z_i, z_i) + g(w_i, w_i).$$

Thus, $F_{*p}z_i = -F_{*p}w_i$ and $g(z_i, z_i) = -g(w_i, w_i),\ i = 1, 2, \ldots, k$. Hence,

$$\begin{aligned}\|F_{*p}\|^2 &= \sum_{i=1}^{t} g(e_i,\ e_i)h(F_{*p}e_i,\ F_{*p}e_i)\\ &= \sum_{i=1}^{t} g_{/S(H)}(\pi_H(e_i),\ \pi_H(e_i))h_{/S(ImF_{*p})}(F^S_{*p}\pi_H(e_i),\ F^S_{*p}\pi_H(e_i))\\ &= \|F^S_{*p}\|^2,\end{aligned}$$

which proves the required equality. □

We recall here the definition of the semi-Riemannian map (see [10], page 85) which uses quotient spaces:

Definition 3.5. Let $f : (M, g) \to (N, h)$ be a map between semi-Riemannian manifolds. Then f is called semi-Riemannian at $p \in M$ if

$$\overline{f}_{*p} : (\overline{H}(p), g_{/\overline{H}(p)}) \to (\overline{A}_2(p), h_{/\overline{A}_2(p)})$$

is an (into) isometry, where $(\overline{H}(p), g_{/\overline{H}(p)})$ and $(\overline{A}_2(p), h_{/\overline{A}_2(p)})$ are the quotient inner product spaces given by:

$$\begin{aligned}\overline{H}(p) &= H_p/Rad(V),\\ \overline{A}_2(p) &= Imf_{*p}/Rad(Imf_{*p})\end{aligned}$$

and $\overline{f}_{*p}$ is the quotient of f_{*p}. Moreover, f is called semi-Riemannian if f is semi-Riemannian at each $p \in M$.

We may characterize the above notion as follows:

Proposition 3.6. *A map $F : (M, g) \to (N, h)$ between semi-Riemannian manifolds is semi-Riemannian at $p \in M$ if and only if F_{*p} preserves the inner products on the screen horizontal vectors, that is, the screen tangent map F^S_{*p} is an (into) isometry map. Moreover, F is a semi-Riemannian map if and only if F is semi-Riemannian at each $p \in M$.*

Remark 3.7. An (into) isometry map F^S_* remains an (into) isometry when one changes the screen distribution, since this fact can easily be justified by using basis. Hence, it follows that the above Proposition 3.6 is independent of the screen distribution chosen and therefore this notion is well defined.

The screen tangent map F^S_* satisfies the following:

Lemma 3.8. *Let $F : (M^m, g) \to (N^n, h)$ be a semi-Riemannian map between semi-Riemannian manifolds. Then:*

1. *$rankF^S_* = rankF - \dim Rad(V) = \dim S(H)$;*
2. *$\|F_*\|^2 = rankF^S_*$;*

3. *in the particular case, when* (M, g) *and* (N, h) *are Riemannian manifolds, the semi-Riemannian map* F *becomes a Riemannian map, defined by Fischer, in* [7].

Let us recall from [2] that a C^1 map $F : (M^m, g) \to (N^n, h)$ between semi-Riemannian manifolds is called horizontally weakly conformal at $p \in M$ with square dilation $\Lambda(p)$ if

$$g({}^*F_{*p}U, {}^*F_{*p}U) = \Lambda(p)h(U, V), \quad \text{for any} \quad U, V \in T_{F(p)}N,$$

for some $\Lambda(p) \in \mathbb{R}$; it is said to be horizontally weakly conformal (on M) if it is horizontally weakly conformal at every point $p \in M$.

Note that under the condition $\Lambda(p) \in \mathbb{R}\backslash\{0\}$, we obtain that F is horizontally conformal at $p \in M$. Moreover, we say that F is horizontally homothetic if F is horizontally conformal on M and the square dilation $\Lambda : M \to \mathbb{R}\backslash\{0\}$ is constant.

4. Definition and Examples of Conformal Semi-Riemannian Maps

Now, we introduce the main notion of the present paper, which generalizes the above notions of Riemannian and semi-Riemannian maps.

Definition 4.1. Let $F : (M^m, g) \to (N^n, h)$ be a smooth map between semi-Riemannian manifolds.

(i) We say that F is conformal semi-Riemannian at $p \in M$ if $0 < rankF \leq min\{m, n\}$ and the screen tangent map F^S_{*p} is conformal, that is, there exists a non-zero real number $\Lambda(p)$ (called square dilation) such that:

$$F^S_{*p} = F_{*p/S(H_p)} : (S(H_p), g_{/S(H_p)}) \to (S(ImF_{*p}), h_{/S(ImF_{*p})})$$

satisfies:

$$h_{F(p)}(F_{*p}X, F_{*p}Y) = \Lambda(p)g_p(X, Y), \quad \text{for any } X, Y \in S(H_p). \tag{4.1}$$

(ii) Moreover, we call F a conformal semi-Riemannian map if F is conformal semi-Riemannian at each $p \in M$.

(iii) In particular, if a conformal semi-Riemannian map F is of constant square dilation, we call it homothetic semi-Riemannian.

Remark 4.2. (i) In the above definition, the square dilation $\Lambda(p)$ is non-zero, since the image of F^S_{*p} is nondegenerate with respect to h. In the particular case when F is a map between Riemannian manifolds, $\Lambda(p)$ is positive.

(ii) As F is conformal semi-Riemannian at p (that is, F^S_{*p} is a conformal map with $\Lambda(p) \neq 0$), it follows that F^S_{*p} is injective. Indeed, if we suppose that there exist $X, Y \in S(H_p)$ such that:

$$F^S_{*p}(X) = F^S_{*p}(Y), \tag{4.2}$$

then the following assertions are equivalent to (4.2):

$F_{*p}X = F_{*p}Y$;
$h_{F(p)}(F_{*p}X, Z) = h_{F(p)}(F_{*p}Y, Z)$, for any $Z \in ImF_{*p} = \{F_{*p}U | U \in S(H_p)\}$;
$h_{F(p)}(F_{*p}X, F_{*p}U) = h_{F(p)}(F_{*p}Y, F_{*p}U)$, for any $U \in S(H_p)$;
$\Lambda(p)g(X, U) = \Lambda(p)g(Y, U)$, for any $U \in S(H_p)$.

Since $\Lambda(p) \neq 0$ and $S(H_p)$ is nondegenerate, then $X = Y$.
Now, we provide some classes of examples in the semi-Riemannian context:

Example. Let $\mathbb{R}^n_q$ denote the semi-Euclidean space of dimension n and index q, endowed with the inner product:

$$g(x, y) = -x_1y_1 - \cdots - x_qy_q + x_{q+1}y_{q+1} + \cdots + x_ny_n,$$

for any $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in \mathbb{R}^n_q$.

We construct here a map

$$F : \mathbb{R}^3_1 \to \mathbb{R}^4_2$$

between the semi-Euclidean spaces $(\mathbb{R}^3_1, g)$ and $(\mathbb{R}^4_2, h)$ defined for any $a, b \in \mathbb{R}$, by

$$F(x_1, x_2, x_3) = (e^{x_2} sinx_3, e^{x_2} cosx_3, a, b).$$

Then we have

$$V = kerF_* = span\{\partial x_1\}$$

and

$$H = (kerF_*)^{\perp} = span\{\partial x_2, \partial x_3\}.$$

It follows that $rankF = 2$ and we get

$$F_*\partial x_2 = e^{x_2} sinx_3 \partial y_1 + e^{x_2} cosx_3 \partial y_2$$
$$F_*\partial x_3 = e^{x_2} cosx_3 \partial y_1 - e^{x_2} sinx_3 \partial y_2.$$

Therefore we have:

$$h(F_*\partial x_k, F_*\partial x_k) = -e^{2x_2} g(\partial x_k, \partial x_k), \quad k = 2, 3.$$

We conclude that F is a conformal semi-Riemannian map with the square dilation, $\Lambda : \mathbb{R}^3 \to \mathbb{R}$, defined by $\Lambda(x_1, x_2, x_3) = -e^{2x_2}$.

Some classes of conformal semi-Riemannian maps between semi-Riemannian manifolds can be obtained by the next statements, namely Propositions 4.3, Proposition 4.4 and Corollary 4.5:

Proposition 4.3. *Let $F : (M, g) \to (N, h)$ be a map between semi-Riemannian manifolds. Then, F is horizontally weakly conformal (see* [2], page 444*) with non-zero square dilation if and only if F is conformal semi-Riemannian with $ImF_* = TN$ and $Rad(V) = \{0\}$.*

Proposition 4.4. *Let $F : (M, g) \to (N, h)$ be a semi-Riemannian map between semi-Riemannian manifolds. Then, F is a conformal semi-Riemannian map with the square dilation $\Lambda = 1$.*

Proof. It is easy to see that we can identify by linear isometry the quotient spaces constructed in [10] at each point $p \in M$, as follows:

$$\begin{aligned} L_1 &\cong Rad(V);\\ A_1 &\cong V + H = [Rad(V)]^{\perp};\\ A_2 &\cong ImF_*;\\ L_2 &\cong ImF_* \cap (ImF_*)^{\perp} = Rad(ImF_*);\\ \overline{H} &= H/L_1 \cong S(H);\\ \overline{A}_2 &= A_2/L_2 \cong S(ImF_*). \end{aligned}$$

Then we complete the proof in a straightforward way. □

Corollary 4.5. *Let $F : (M, g) \to (N, h)$ be a map between semi-Riemannian manifolds.*

(a) *ImF is an isometric immersed submanifold (see* [18], page 121*) in N if and only if F is a conformal semi-Riemannian map with $KerF_* = \{0\}$ and $\Lambda = 1$.*
(b) *F is a semi-Riemannian submersion (see* [6]*) if and only if F is a conformal semi-Riemannian map with $ImF_* = TN$, $Rad(V) = \{0\}$ and $\Lambda = 1$.*
(c) *F is a horizontally weakly conformal map of square dilation $\Lambda = 1$ if and only if it is a conformal semi-Riemannian map with $ImF_* = TN$ and $\Lambda = 1$. We note that a semi-Riemannian submersion is a horizontally weakly conformal map (see* [2], page 444*) with the square dilation $\Lambda = 1$.*

Example. (in Riemannian context): Let $F : M \to N$ be a conformal Riemannian map between Riemannian manifolds, with dilation λ, defined by Şahin in [19]. Then F provides an example of a conformal semi-Riemannian map with positive square dilation $\Lambda = \lambda^2 : M \to \mathbb{R}\backslash\{0\}$.

5. Generalized Eikonal Equation

Eikonal equations are an interesting topic for both PDE and differential geometry (see [10,15,19]). We provide here a generalized eikonal equation which states a relation between the square norm of the tangent map and the non-degenerate rank of a conformal semi-Riemannian map.

Proposition 5.1. *Let $F : (M^m, g) \to (N^n, h)$ be a map between semi-Riemannian manifolds which is a conformal semi-Riemannian map at $p \in M$ with $\Lambda(p) \neq 0$.*

Then:

$$\|F_{*p}\|^2 = \Lambda(p) rank F^S_{*p}. \tag{5.1}$$

Proof. We have that the map

$$F^S_{*p} : (S(H_p), g_{/S(H_p)}) \to (S(ImF_{*p}), h_{/S(ImF_{*p})})$$

is conformal and let

$${}^*F^S_{*p} : (S(ImF_{*p}), h_{/S(ImF_{*p})}) \to (S(H_p), g_{/S(H_p)})$$

be the adjoint of F^S_{*p}. Then:

$$g_{/S(H_p)}(^*F^S_{*p} \circ F^S_{*p}u, v) = h_{/S(ImF_{*p})}(F^S_{*p}u, F^S_{*p}v) \\ = \Lambda(p) g_{/S(H_p)}(u, v), \quad \forall u, v \in S(H_p). \tag{5.2}$$

From Remark 4.2(ii), we have:

$$rank F^S_{*p} = \dim S(H_p) = \dim H_p - \dim Rad(V_p). \tag{5.3}$$

Hence, by applying consequently Lemma 3.4 and the relations (5.2) and (5.3), one has:

$$\|F_*\|^2 = \| \ F^S_{*p}\|^2 = trace_g {}^*F^S_{*p} \circ F^S_{*p} = \sum_{i=1}^{t} \varepsilon_i \ g_{/S(H_p)}(^*F^S_{*p} \circ F^S_{*p} \ e_i, e_i) \\ = \Lambda(p) \sum_{i=1}^{t} \varepsilon_i \ g_{/S(H_p)}(e_i, e_i) = \Lambda(p) \ \dim S(H_p) \\ = \Lambda(p) \mathrm{rank} \ F^S_{*p},$$

where $\{e_1, \ldots, e_t\}$ is an orthonormal basis (with respect to g) of the nondegenerate screen horizontal distribution $S(H)$ and $\varepsilon_i = g(e_i, e_i) \in \{-1, 1\}$, $i = 1, \ldots, t$, which complete the proof. □

Remark 5.2. (i) The statements of Lemma 3.4 and Proposition 5.1 are independent of the screen horizontal distribution which was chosen in the proof.

(ii) When F is a homothetic semi-Riemannian map, then the right hand side of the relation (5.1) is constant on each connected component of M, since the map $\|F_*\|^2 : M \to \mathbb{R}$, defined by $\|F_*\|^2(p) = \|F_{*p}\|^2$, is a continuous function.

(iii) From the above remark, it follows that F is a solution of the generalized eikonal equation, provided that F is a homothetic semi-Riemannian map.

Let $F : (M^m, g) \to (N^n, h)$ be a smooth map between semi-Riemannian manifolds. For any $p \in M$, let us define the linear transformation

$$Q_p : S(ImF_{*p}) \to S(ImF_{*p}), \ by \\ Q_p = F^S_{*p} \circ^* F^S_{*p}$$

to obtain the following characterization of conformal semi-Riemannian map.

Theorem 5.3. *A smooth map $F : (M^m, g) \to (N^n, h)$ is conformal semi-Riemannian if and only if for any $p \in M$, there exists a smooth function*

$$\Lambda : M \to \mathbb{R},$$

such that

$$Q_p^2 = \Lambda(p) Q_p. \tag{5.4}$$

Proof. We have that the relation (5.4) is equivalent to

$$Q_p^2 W = \Lambda(p) Q_p(W), \quad \text{for any } W \in S(ImF_{*p}). \tag{5.5}$$

Since $S(ImF_{*p})$ is nondegenerate, then (5.5) is equivalent to each of the following assertions

$$\begin{aligned} &h_{F(p)}(U, Q_p^2 W) = h_{F(p)}(U, Q_p^2 W), \text{ for any } U, W \in S(ImF_{*p}); \\ &h_{F(p)}(U, F_{*p}^S \circ^* F_{*p}^S \circ F_{*p}^S \circ^* F_{*p}^S W) = \Lambda(p) h_{F(p)}(U, F_{*p}^S \circ^* F_{*p}^S W), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } U, W \in S(ImF_{*p}); \\ &g_p({}^*F_{*p}^S U, {}^* F_{*p}^S \circ F_{*p}^S \circ^* F_{*p}^S W) = \Lambda(p) g_p({}^*F_{*p}^S U, {}^* F_{*p}^S W), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } U, W \in S(ImF_{*p}); \\ &h_{F(p)}(F_{*p}^S \circ^* F_{*p}^S U, F_{*p}^S \circ^* F_{*p}^S W) = \Lambda(p) g_p({}^*F_{*p}^S U, {}^* F_{*p}^S W), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } U, W \in S(ImF_{*p}). \end{aligned} \tag{5.6}$$

The direct statement follows immediately, since if F is a conformal semi-Riemannian map at p, then the last equality of (5.6) is satisfied.

Conversely, if we suppose that the relation (5.4) is true, then by the above equivalence, the last equality of the relation (5.6) is satisfied. We prove that F is a conformal semi-Riemannian map. To do this, we first notice that the map ${}^*F_{*p}^S : S(ImF_{*p}) \to S(H_p)$ is onto. Indeed, the image of ${}^*F_{*p}^S : S(ImF_{*p}) \to S(H_p)$ is $S(H_p)$, since if we suppose otherwise, then there exists a non-zero vector field $\xi \in S(H_p)$, such that $g_p({}^*F_{*p}^S Z, \xi) = 0$, for any $Z \in S(ImF_{*p})$. Therefore we have:

$$0 = g_p({}^*F_{*p}^S Z, \xi) = h_{F(p)}(Z, F_{*p}\xi), \quad \text{for any } Z \in S(ImF_{*p}).$$

Now, as $S(ImF_{*p})$ is nondegenerate with respect to h, $F_{*p}\xi = 0$, that is, $\xi \in Ker(F_{*p}) = V_p$ which is orthogonal to $S(H_p)$. Since one has simultaneously (i) $\xi \in S(H_p)$, (ii) ξ is orthogonal to $S(H_p)$ and (iii) $S(H_p)$ is nondegenerate with respect to g, it follows that $\xi = 0$ which is a contradiction. Therefore, the last equality of (5.6) is equivalent to (4.1), since for any $X, Y \in S(H_p)$, there exist $U, W \in S(ImF_{*p})$ such that $X = {}^*F_{*p}^S U$ and $Y = {}^*F_{*p}^S W$, which shows that F is conformal semi-Riemannian at any point $p \in M$, and the proof is completed. □

Remark 5.4.
1. If (M, g) and (N, h) are Riemannian manifolds, we reobtain Fischer's result [7], that is, F is a Riemannian map if and only if $Q_p = F_{*p} \circ^* F_{*p}$ is a projection of T_pM, i.e. $Q_p^2 = Q_p$.
2. If (M, g) and (N, h) are semi-Riemannian manifolds, then we reobtain Şahin's result [19], that is, F is a conformal semi-Riemannian map if and only if the operator Q_p defined on T_pM by $Q_p = F_{*p} \circ^* F_{*p}$ satisfies the relation (5.4).
3. As noticed in ([10], page 92), Fischer's theorem is not valid when M and N are semi-Riemannian manifolds and when F is a semi-Riemannian map if we take Q_p as an operator of T_pM. To generalize Fischer's result, we stated Theorem 5.3 by taking Q_p defined on a screen distribution $S(ImF_{*p})$.

6. Conformal Semi-Riemannian Maps and Their Harmonicity

Definition 6.1. Let $F : (M^m, g) \to (N^n, h)$ be a smooth map between semi-Riemannian manifolds and let ∇^M and $\nabla^{F^{-1}TN}$ denote, respectively, the

Levi–Civita connection on M and the pull-back connection. Then, F is harmonic if its tension field $\tau(F)$ vanishes identically, that is,

$$\tau(F) = trace_g(\nabla_{\boldsymbol{\cdot}} F_{*\boldsymbol{\cdot}}) = \sum_{i=1}^{m} (\nabla F_*)(e_i, e_i) = 0,$$

where $\{e_i\}_{i=1,\dots,m}$ is an orthonormal frame on M and the second fundamental form ∇F_* of F is given by

$$(\nabla F_*)(X, Y) = \nabla_X^{F^{-1}(TN)} F_* Y - F_*(\nabla_X^M Y), \quad \text{for any } X, Y \in \Gamma(TM).$$

For the harmonicity of other structures on manifolds, we refer to [4]. We recall now from [2] a geometric notion related to harmonicity, namely the following :

Definition 6.2. Let $F : (M, g) \to (N, h)$ be a C^2 map between semi-Riemannian manifolds. Then F is a harmonic morphism if, for any C^2 harmonic function f defined on an open subset $\overline{N}$ of N with $F^{-1}(\overline{N})$ non-empty, the composition $f \circ F$ is harmonic on $F^{-1}(\overline{N})$.

The above notion was characterized in both [9] and [13] by the following:

Theorem 6.3. *A C^2 map between semi-Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally weakly conformal.*

If D is a nondegenerate differentiable distribution of rank k on a semi-Riemannian manifold (M, g) with the Levi–Civita connection ∇, then TM splits into the direct sum $TM = D \oplus D^\perp$, where $D^\perp$ is the orthogonal distribution of D with respect to g. Moreover, D is called minimal if at each $p \in M$, the mean curvature field $\mu(D) \in \Gamma(F^{-1}TN)$ of D vanishes, i.e.

$$\mu(D) = \frac{1}{k} trace_g(\nabla_{\boldsymbol{\cdot}} {\boldsymbol{\cdot}})^\perp = \frac{1}{k} \sum_{i=1}^{k} g(e_i, e_i)(\nabla_{e_i} e_i)^\perp = 0,$$

where $(\nabla_{e_i} e_i)^\perp$ denotes the component of $\nabla_{e_i} e_i$ in the orthonormal complementary distribution $D^\perp$ on M and $\{e_i\}_{i=1,\dots,k}$ is an orthonormal basis of D.

When the distribution D is integrable, then D is minimal if and only if any leaf of D is a minimal submanifold of M. (For degenerate distributions we refer the reader to [3]).

Then the calculation in the semi-Riemannian context follows the same steps as in the Riemannian case (see [19]) and, consequently, Theorem 4.1 from [19] is now valid in the semi-Riemannian case, as follows:

Theorem 6.4. *Let $F : (M^m, g) \to (N^n, h)$ be a non-constant proper conformal semi-Riemannian map between semi-Riemannian manifolds, such that the vertical distribution is nondegenerate and of codimension greater than 2. Then any three conditions imply the fourth one:*

(i) *F is harmonic;*
(ii) *F is horizontally homothetic;*
(iii) *the vertical distribution is minimal;*
(iv) *the distribution ImF_* is minimal.*

In view of Definition 4.1, we note that in the semi-Riemannian context, both minimal immersions and harmonic morphisms are particular classes of harmonic maps which are conformal semi-Riemannian and hence both these classes can be studied in a unitary manner.

Acknowledgements

Both authors commemorate Professor Demir N. Kupeli who passed away in 2015. The second author was supported by the Scientific and Technical Research Council of Turkey (TUBITAK) for Advanced Fellowships Programme.

References

[1] Atçeken, M., Kılıç, E.: Semi-invariant lightlike submanifolds of semi-Riemannian product manifold. Kodai Math. J. **30**, 361–378 (2007)

[2] Baird, P., Wood, J.C.: Harmonic Morphisms Between Riemannian Manifolds. Clarendon Press, Oxford (2003)

[3] Bejan, C.L., Duggal, K.L.: Global lightlike manifolds and harmonicity. Kodai Math. J. **28**, 131–145 (2005)

[4] Bejan, C.L., Druta-Romaniuc, S.L.: Harmonic almost complex structures with respect to the general natural metrics. Meditarr. J. Math. **11**(1), 123–136 (2014)

[5] Duggal, K.L., Bejancu, A.: Lightlike Submanifolds of Semi-Riemannian Manifolds and Their Applicaitons. Kluwer Academic, Dordrecht (1996)

[6] Falcitelli, M., Ianus, S., Pastore, A.M.: Riemannian Submersions and Related Topics. World Scientific, Singapore (2004)

[7] Fischer, A.E.: Riemannian maps between Riemannian manifolds. Contemp. Math. **132**, 331–336 (1992)

[8] Fuglede, B.: Harmonic morphisms between Riemannian manifolds. Ann. Inst. Fourier (Grenoble) **28**, 107–144 (1978)

[9] Fuglede, B.: Harmonic morphisms between semi-Riemannian manifolds. Ann. Acad. Sci. Fen. Math. **21**, 31–50 (1996)

[10] García-Río, E., Kupeli, D.N.: Semi-Riemannian Maps and Their Applications. Kluwer Academic, Dordrecht (1999)

[11] Gray, A.: Pseudo-Riemannian almost product manifolds and submersions. J. Math. Mech. **16**, 715–737 (1967)

[12] Gu, X., Wang, Y., Yau, S.T.: Coperiodic matrices. Commun. Inf. Syst. **3**(3), 153–169 (2003)

[13] Ishihara, T.: A mapping of Riemannian manifolds which preserves harmonic functions. J. Math. Kyoto Univ. **19**, 215–229 (1979)

[14] Kock, A.: A geometric theory of harmonic and semi-conformal maps. Cent. Eur. J. Math. **2**(5), 708–724 (2004)

[15] Kupeli, D.N.: The eikonal equation of an indefinite metric. Acta Appl. Math. **40**, 245–253 (1995)

[16] Miao, J., Wang, Y., Gu, X., Yau, S.T.: Optimal global conformal surface parametrization for visualization. Commun. Inf. Syst. **4**(2), 117–134 (2005)

[17] Mustafa, M.T.: Applications of harmonic morphisms to gravity. J. Math. Phys. **41**(10), 6918–6929 (2000)

[18] O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York (1983)

[19] Şahin, B.: Conformal Riemannian maps between Riemannian manifolds, their harmonicty and decomposition theorems. Acta Appl. Math. **109**, 829–847 (2010)

[20] Wang, Y., Gu, X., Chan, T.F., Thompson, P.M., Yau, S.T.: Brain surface conformal parametrization with the Ricci flow. In: IEEE International Symposium on Biomedical ImagingFrom Nano to Macro (ISBI), pp. 1312–1315, Washington, DC (2007)

[21] Wang, Y., Yin, X., Zhang, J., Gu, X., Chan, T.F., Thompson, P.M., Yau, S.T.: In: Brain Mapping with the Ricci Flow Conformal Parameterization and Multivariate Statistics on Deformation Tensors, 2nd MICCAI Workshop on Mathematical Foundations of Computational Anatomy, pp. 36–47, New York (2008)

Cornelia-Livia Bejan
Univ. Tehnica "Gh. Asachi" Iasi
Iasi, Romania

and

Seminar Matematic
Universitatea "Al. I. Cuza" Iasi
Bd. Carol I no. 11, 700506 Iasi, Romania
e-mail: bejanliv@yahoo.com
http://math.etc.tuiasi.ro/bejan/

and

Şemsi Eken
Faculty of Science
Karadeniz Technical University
Trabzon, Turkey
e-mail: semsieken@hotmail.com

Received: May 28, 2015.
Accepted: July 6, 2015.