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Einstein Metrics Induced by Natural Riemann Extensions

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Abstract. Clifford algebras are used in theoretical physics and in particular, in the general theory of relativity, where Einstein's equations are rewritten in Girard (Adv Appl Clifford Algebras 9(2):225–230, 1999) within a Clifford algebra. Let M be a manifold with a torsion-free connection which induces on its cotangent bundle T^*M, a semi-Riemannian metric $\bar{g}$, called the natural Riemann extension, Kowalski and Sekizawa (Publ Math Debrecen 78:709–721, 2011). The main result of the present paper gives a necessary and sufficient condition for $\bar{g}$ restricted to certain hypersurfaces of T^*M to be Einstein.

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1. Introduction

Ricci curvature plays a fundamental role in general relativity, especially in the Einstein field equations. The existence or the non-existence of Einstein metrics on a manifold is related to some Clifford algebras, as shown in many papers. A few examples are pointed out in what follows.

Throughout this note, by a positive Einstein metric we mean an Einstein metric with positive scalar curvature.

The use of Weitzenböck formula for Dirac operators yields to several examples of manifolds of dimension ≥5, which do not admit any positive Einstein metric. Moreover, a $K3$ surface (from Kodaira's classification) is a complex surface with vanishing first Chern class and no global holomorphic one-forms. This spin surface admits no metric with positive scalar curvature

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and in particular no positive Einstein metric. The obstructions for the existence of positive Einstein metrics on a manifold M are related with the Clifford algebra attached to the tangent bundle of M, (see [2, pp. 169]).

On the other hand, Burdujan introduced in [4,5] some Clifford–Kähler manifolds which are proved to be Einstein. Also, the existence of parallel non-flat even Clifford structures of rank ≥ 3 on a complete simply connected Riemannian manifold M^n assures that the manifold is Einstein, provided $r \neq 4$, $n \neq 8$, where r is the rank of the Clifford structure and n is the dimension of the manifold M^n (see [13, Proposition 2.10]).

The purpose of this note is to construct a family of examples of Einstein manifolds with positive scalar curvature.

First, we note that a semi-Riemannian metric on a manifold M gives rise to a natural vector bundle isomorphism between the Clifford bundle of M and the exterior bundle of M

$$Cl(T^*M) = \Lambda(T^*M),$$

which is induced by the corresponding isomorphism on each fiber. This is why one can view the sections of the Clifford bundle as differential forms on M.

Patterson and Walker introduced in [14] a semi-Riemannian metric g on the total space T^*M of the cotangent bundle of a manifold M^n endowed with a symmetric connection ∇, (see also [16,17]). Since then, this metric g, called a Riemann extension, was extensively used by several authors for different purposes. For instance, this metric was used recently by [3,7,8] in context with Einstein structures, respectively Ricci solitons. (The concept of Ricci solitons extends in a certain way the Einstein metric). In [11,15], the notion of Riemann extension was generalized by Kowalski and Sekizawa to the notion of natural Riemann extension $\bar{g}$ (see the relations (4)), which is a semi-Riemannian metric of neutral signature (n, n). Certain metrics of neutral signature were studied by Crasmareanu and Piscoran [6] in relation with Clifford algebras. In particular, when M is a surface, we note that $(T^*M, \bar{g})$ is an example of a four-dimensional non-Lorentzian, semi-Riemannian manifold. The importance of neutral metrics on four-dimensional manifolds is emphasized in [10]. In our paper, if we take M to be a surface, then the family of hypersurfaces of $(T^*M, \bar{g})$ are Lorentz manifolds.

In Sect. 2, we recall some notions and results and also some constructions and calculations are included for later use. In Sect. 3, we prove some properties of a family $\{H_t/t \in \mathbb{R} - \{0\}\}$ of non-degenerate hypersurfaces of $(T^*M, \bar{g})$ and we obtain some important consequences. Our main result here states that each hypersurface H_t of the above family is Einstein, with positive scalar curvature. Some comments are provided at the end.

2. Natural Riemann Extension

The notations used here are taken from [11].

If M is an n-dimensional manifold ($n \geq 2$), then the space of phases (which is its cotangent bundle) T^*M contains all pairs (x, w), with $x \in M$

and $w \in T_x^*M$. Let $p : T^*M \to M,\ p(x,w) = x$, be the natural projection of T^*M to M. For any local coordinate chart $(U; x^1, \dots, x^n)$ on M, we denote by $(p^{-1}(U); x^1, \dots, x^n, x^{1*}, \dots, x^{n*})$ the corresponding coordinate chart on T^*M such that for any $i = \overline{1,n}$, the function $x^i \circ p$ on $p^{-1}(U)$ is identified with x^i on U and we have $x^{i*} = w_i = w((\frac{\partial}{\partial x^i})_x)$ at any point $(x,w) \in p^{-1}(U)$. With the notation $\frac{\partial}{\partial x^i} = \partial_i$ and $\frac{\partial}{\partial w^i} = \partial_{i*}, i = \overline{1,n}$, at each point $(x,w) \in T^*M$ one has a basis

$$\{(\partial_1)_{(x,w)}, \dots, (\partial_n)_{(x,w)}, (\partial_{1*})_{(x,w)}, \dots, (\partial_{n*})_{(x,w)}\}$$

for the tangent space $(T^*M)_{(x,w)}$.

The Liouville type vector field $\mathbf{W}$, globally defined on T^*M is expressed in local coordinates by

$$W = \sum_{i=1}^{n} w_i \partial_{i*}.$$

Everywhere here, $\mathcal{F}(M)$ and $\chi(M)$ will denote the set of all smooth real functions on M and respectively the set of all vector fields on M.

The vertical lift f^v on T^*M of a function $f \in \mathcal{F}(M)$, is defined by $f^v = f \circ p$.

The vertical lift X^v on T^*M of a vector field $X \in \chi(M)$ is a function (called evaluation function) defined by

$$X^v(x,w) = w(X_x),$$

or equivalently $X^v(x,w) = w_i X^i(x)$, where $X = X^i \partial_i$.

Remark 1. We stress that the vertical lift of any vector field on M is a function on T^*M and not a vector field tangent to T^*M .

A vector field $U \in \chi(T^*M)$ on the total space of the cotangent bundle of M is defined by its action on all evaluation functions. More precisely, we recall the following:

Proposition 2. [17] *Let U_1 and U_2 be vector fields on T^*M. If $U_1(Z^v) = U_2(Z^v)$ holds for all $Z \in \chi(M)$, then $U_1 = U_2$.*

We use here some constructions of lifts from [18].
Any 1-form $\alpha \in \Omega^1(M)$ on M can be lifted to a vector field α^v tangent to T^*M, which is defined by

$$\alpha^v(Z^v) = (\alpha(Z))^v, \quad \forall Z \in \chi(M),$$

or equivalently by

$$\alpha^v = \alpha_i \partial_{i*},$$

where $\alpha = \alpha_i dx_i$ and we identify f^v with f when $f \in \mathcal{F}(M)$. It follows that $\alpha^v(f^v) = 0, \forall f \in \mathcal{F}(M)$. The complete lift of a vector field $X \in \chi(M)$ is a vector field $X^c \in \chi(T^*M)$, defined at any point $(x,w) \in T^*M$ by

$$X^c_{(x,w)} = \xi^i(x)(\partial_i)_{(x,w)} - w_h(\partial_i \xi^h)(x)(\partial_{i*})_{(x,w)}, \tag{1}$$

where $X = X^i \partial_i$. Therefore, we have

$$X^c(Z^v) = [X,Z]^v, \forall Z \in \chi(M) \tag{2}$$

and

$$X^c f^v = (Xf)^v, \quad \forall f \in \mathcal{F}(M).$$

Hypothesis If not otherwise stated, we assume the n-dimensional manifold M is endowed with both a symmetric linear connection ∇ (i.e. ∇ is torsion-free) and with a globally defined nowhere zero vector field ξ, which is parallel with respect to ∇, that is

$$\nabla \xi = 0. \tag{3}$$

The symmetric linear connection ∇ on M defines a semi-Riemannian metric on the total space of T^*M, as it was constructed by Sekizawa in [15] by:

$$\begin{aligned} \bar{g}(X^c, Y^c) &= -aw(\nabla_{X_x} Y + \nabla_{Y_x} X) + bw(X_x)w(Y_x); \\ \bar{g}(X^c, \alpha^v) &= a\alpha_x(X_x); \\ \bar{g}(\alpha^v, \beta^v) &= 0, \end{aligned} \tag{4}$$

for all vector fields X, Y and all differential 1-forms α, β on M, where, a, b are arbitrary constants and we may assume $a > 0$ without loss of generality. In the particular case, when $a = 1$ and $b = 0$, we obtain the notion of the classical Riemann extension defined by Patterson and Walker, see [14,16].

Definition 3. The semi-Riemannian metric given in (4) is called a natural Riemann extension, [11,15]. When $b \neq 0$ we call it proper natural Riemann extension.

Let (x, w) be an arbitrary fixed point in T^*M, with $w \neq 0$. We denote $\alpha_1 = w$ and we take a basis $\{\alpha_1, \ldots, \alpha_n\}$ in T_x^*M and then we consider $\{e_1, \ldots, e_n\}$ its dual by basis in T_xM. As in [11], we denote by the same letter the parallel extension of each e_i (along geodesics starting at x) to a normal neighbourhood of x in M, $i = \overline{1, n}$. It follows that $\{e_1, \ldots, e_n\}$ is a local frame on M which is defined around x and satisfies the condition

$$(\nabla_{e_i} e_j)_x = 0, \quad i, j = \overline{1, n}, \tag{5}$$

from which we obtain

$$\bar{g}_{(x,w)}(e_i^c, e_j^c) = bw(e_{i,x})w(e_{j,x}), \quad i, j = \overline{1, n}. \tag{6}$$

Next, we denote by the same letter $\{\alpha_1, \ldots, \alpha_n\}$ the local frame defined around x, which is dual to the local frame $\{e_1, \ldots, e_n\}$ which means

$$\alpha_i(e_j) = \delta_{ij}, \quad i, j = \overline{1, n}. \tag{7}$$

Obviously, $\alpha_{1,x} = w$. Different from the locally orthonormal basis constructed in [11], we constructed in [1] the following orthonormal frame $\{E_i, E_{i*} : i = \overline{1, n}\}$:

$$\begin{aligned} E_1 &= e_1^c + \frac{1-b}{2a}\alpha_1^v; \quad E_{1*} = e_1^c - \frac{1+b}{2a}\alpha_1^v; \\ E_k &= \frac{1}{\sqrt{2a}}(e_k^c + \alpha_k^v); \quad E_{k*} = \frac{1}{\sqrt{2a}}(e_k^c - \alpha_k^v), \quad k = \overline{2, n}. \end{aligned} \tag{8}$$

Remark 4. We note that $\bar{g}(E_j, E_j) = 1$ and $\bar{g}(E_{j*}, E_{j*}) = -1$, $j = \overline{1, n}$. Hence $\bar{g}$ is of neutral signature (n, n).

We recall here that the gradient of a real function $F : N \to \mathbb{R}$ on a (semi-) Riemannian manifold (N, h) is given by $h(gradF, X) = dF(X)$, $\forall X \in \chi(N)$.

The following conventions and formulas will be used later on.

If T is a (1,1)-tensor field on a manifold N, then the contracted vector field $C(T) \in \chi(T^*M)$ is defined at any point $(x, w) \in T^*M$, by its value on any evaluation function as follows:

$$C(T)(X^v)_{(x,w)} = (TX)^v_{(x,w)} = w((TX)_x), \quad \forall X \in \chi(M).$$

For the Levi–Civita connection $\bar{\nabla}$ of the Riemann extension $\bar{g}$, we get the formulas (see e.g., [11]):

$$\begin{aligned}
(\bar{\nabla}_{X^c} Y^c)_{(x,w)} &= (\nabla_X Y)^c_{(x,w)} + C_w((\nabla X)(\nabla Y) + (\nabla Y)(\nabla X))_{(x,w)} \\
&\quad + C_w(R_x(., X)Y + R_x(., Y)X)_{(x,w)} \\
&\quad - \frac{c}{2}\{w(Y)X^c w(X)Y^c + 2w(Y)C_w(\nabla X) + 2w(X)C_w(\nabla Y) \\
&\quad + w(\nabla_X Y + \nabla_Y X)W\}_{(x,w)} + c^2 w(X)w(Y)W_{(x,w)}, \\
(\bar{\nabla}_{X^c} \beta^v)_{(x,w)} &= (\nabla_X \beta)^v_{(x,w)} + \frac{c}{2}\{w(X)\beta^v + \beta(X)W\}_{(x,w)}, \\
(\bar{\nabla}_{\alpha^v} Y^c)_{(x,w)} &= -(i_\alpha(\nabla Y))^v_{(x,w)} + \frac{c}{2}\{w(Y)\alpha^v + \alpha(Y)W\}_{(x,w)}, \\
(\bar{\nabla}_{\alpha^v} \beta^v)_{(x,w)} &= 0, \quad \forall X, Y \in \chi(M), \quad \forall \alpha, \beta \in \Omega^1(M).
\end{aligned} \tag{9}$$

Here, c denotes the fraction $\frac{b}{a}$. Further, for a (1,1)-tensor field T and a 1-form α on M, $i_\alpha(T)$ is a 1-form of M defined by

$$(i_\alpha(T))(X) = \alpha(TX), \quad \forall X \in \chi(M).$$

From [11], we have the following:

$$\begin{aligned}
\bar{R}_{(x,w)}(X^c, Y^c)Z^c &= (R(X,Y)Z)^c_{(x,w)} + \frac{c}{2}\{(w(\nabla_Z Y) + \frac{c}{2}w(Y)w(Z))X^c \\
&\quad - (w(\nabla_Z X) + \frac{c}{2}w(X)w(Z))Y^c - w([X,Y])Z^c\}_{(x,w)} \\
&\quad + C_w((\nabla_X R)(\cdot, Y)Z + (\nabla_X R)(\cdot, Z)Y \\
&\quad - (\nabla_Y R)(\cdot, X)Z - (\nabla_Y R)(\cdot, Z)X \\
&\quad - [\nabla X, R(\cdot, Z)Y] + [\nabla Y, R(\cdot, Z)X] \\
&\quad - (\nabla Z)R(\cdot, X)Y + (\nabla Z)R(\cdot, Y)X \\
&\quad - (R(\cdot, X)Y)(\nabla Z) + (R(\cdot, Y)X)(\nabla Z))_{(x,w)} \\
&\quad + cw(X_x)C_w(R(\cdot, Z)Y - (\nabla Y)(\nabla Z) - (\nabla Z)(\nabla Y))_{(x,w)} \\
&\quad - cw(Y_x)C_w(R(\cdot, Z)X - (\nabla X)(\nabla Z) - (\nabla Z)(\nabla X))_{(x,w)} \\
&\quad - cw(Z_x)C_w(R(X, \cdot)Y - R(Y, \cdot)X - 2[\nabla X, \nabla Y])_{(x,w)} \\
&\quad - \frac{c}{2}\{w(X)w([Y,Z]) - w(Y)w([X,Z]) \\
&\quad - 2w([X,Y])w(Z)\}\mathbf{W}_{(x,w)},
\end{aligned} \tag{10}$$

$$\bar{R}_{(x,w)}(X^c, Y^c)\gamma^v = -(i_\gamma(R(X,Y)))^v_{(x,w)} + \frac{c}{2}w([X,Y])\gamma^v_{(x,w)} - \frac{c}{2}\{\gamma(Y)C_w(\nabla X) - \gamma(X)C_w(\nabla Y)\}_{(x,w)} + \frac{c^2}{4}\{\gamma(Y)w(X) - \gamma(X)w(Y)\}\mathbf{W}_{(x,w)}, \quad (11)$$

$$\bar{R}_{(x,w)}(X^c, \beta^v)Z^c = -(i_\beta(R(\cdot,Z)X))^v_{(x,w)} + \frac{c}{2}\{\beta(Z)X^c + \beta(X)Z^c\}_{(x,w)} + \frac{c}{2}\{w(\nabla_X Z) - \frac{c}{2}w(X)w(Z)\}\beta^v_{(x,w)} + \frac{c}{2}\{\beta(Z)C_w(\nabla X) + 2\beta(X)C_w(\nabla Z)\}_{(x,w)} - \frac{c^2}{2}\{\beta(X)w(Z) + \frac{1}{2}\beta(Z)w(X)\}\mathbf{W}_{(x,w)}, \quad (12)$$

$$\bar{R}_{(x,w)}(X^c, \beta^v)\gamma^v = -\frac{c}{2}\{\gamma(X)\beta^v + \beta(X)\gamma^v\}_{(x,w)}, \quad (13)$$

$$\bar{R}_{(x,w)}(\alpha^v, \beta^v)Z^c = 0, \quad (14)$$

$$\bar{R}_{(x,w)}(\alpha^v, \beta^v)\gamma^v = 0, \quad (15)$$

for all $X, Y, Z \in \chi(M)$ and one-forms α, β, γ of M.

For later use, we recall the following formula, obtained in [1]:

Theorem 5. *Let M be a manifold endowed with a symmetric linear connection ∇, which defines the natural Riemann extension $\bar{g}$ on T^*M. Then, the gradient (with respect to $\bar{g}$) for any vertical lift $Z^v \in \mathcal{F}(T^*M)$ of a vector field $Z \in \chi(M)$ is given by:*

$$gradZ^v = \frac{1}{a}\{Z^c + 2C(\nabla Z) - \mathsf{c}Z^v\,\boldsymbol{W}\}, \quad (16)$$

where $\mathcal{F}(T^*M)$ denotes the set of all smooth real functions on T^*M and the contraction C is applied to the (1,1)-tensor field ∇Z on M, defined by $(\nabla Z)(X) = \nabla_X Z$, $\forall X \in \chi(M)$.

3. Hypersurfaces of the Total Space of the Cotangent Bundle

In this section, we assume that a manifold M is endowed with a symmetric linear connection ∇, which induces on the total space of the cotangent bundle T^*M of M the natural Riemann extension $\bar{g}$. We assume $\bar{g}$ is proper (i.e. $b \neq 0$).

The evaluation map $\bar{f} : T^*M \to \mathbb{R}$ is defined by

$$\bar{f} = \xi^v, \quad (17)$$

or equivalently by $\bar{f}(x,w) = w_x(\xi_x)$, for any $(x,w) \in T^*M$.
Let

$$H_t = \bar{f}^{-1}(t) = \{(x,w) \in T^*M / \bar{f}(x,w) = t\},$$

be the hypersurfaces level set in T^*M, endowed with the restriction $g_t = \bar{g}/_{H_t}$ of the natural Riemann extension $\bar{g}$ inherited from T^*M, where $t \in \mathbb{R} - \{0\}$.

Remark 6. For any $t \in \mathbb{R} - \{0\}$, one has:

(i) $H_t \subset T^*M - \{0\}$ (i.e. T^*M without the zero section);

(ii) $grad\bar{f}$ is non-zero (hence orthogonal to H_t), at any point of H_t.

Theorem 7. *Let (M, ∇) be a manifold endowed with a symmetric linear connection inducing the proper natural Riemann extension $\bar{g}$ on T^*M. If $t \in \mathbb{R} - \{0\}$, then:*

(i) *g_t is non-degenerate on H_t, hence (H_t, g_t) is a semi-Riemannian hypersurface of T^*M;*

(ii) *The tangent space $T_{(x,w)}H_t$ at any point $(x, w) \in H_t$ is generated by $\alpha^v + X^c$, where*

$$\alpha \in \Omega^1(M),\ X \in \chi(M) \quad \text{and} \quad \alpha(\xi) + w([X, \xi]) = 0; \tag{18}$$

(iii) *At any point (x, w) of H_t, a vector field normal to H_t is given by*

$$grad\bar{f} = \frac{1}{a}\{\xi^c - \mathsf{c}t\,\boldsymbol{W}\}, \tag{19}$$

where ξ^c denotes the complete lift of ξ;

(iv) *The system $\{\alpha_2^v, \dots, \alpha_n^v, e_1^c, \dots, e_n^c\}$ defined above is a basis of vectors tangent to H_t, at any point (x, w) of H_t.*

Proof. Since $\bar{f}$ is defined above by (17), we use (16) from Theorem 5, and we take into account that ξ is parallel with respect to ∇. Therefore we obtain (iii), since at any point (x, w) of H_t we have $\bar{f}(x, w) = t$. From the definition of the natural Riemann extension, we compute $\bar{g}(grad\bar{f}, grad\bar{f}) = -b[\frac{w(\xi)}{a}]^2$, which shows that at any point $(x, w) \in H_t$, we have $\|grad\bar{f}\|^2 = -b(\frac{t}{a})^2 \neq 0$ and hence, $grad\bar{f}$ is time-like or space-like according as $b > 0$ or $b < 0$, respectively. Consequently, (i) is proved. Next, we note that T^*M is generated by the vertical lifts α^v of 1-forms α on M, together with the complete lifts X^c of vector fields X on M. It follows that the tangent space $T_{(x,w)}H_t$ of H_t at any point $(x, w) \in T^*M$, is generated by the vector fields of the form $\alpha^v + X^c$ which are orthogonal to $grad\bar{f}$ with respect to $\bar{g}$, i.e. they should satisfy the condition

$$\alpha(\xi) \circ p(x, w) + w_x([X, \xi]_x) = 0, \quad \text{for any } (x, w) \in H_t,$$

which is equivalent to (ii). To prove (iv), we first note from (5) that the bracket of any two vector fields from $\{e_1^c, \dots, e_n^c\}$ vanishes at $(x, w) \in H_t$, since ∇ is torsion-free. Then, each vector $e_1^c, \dots, e_n^c$ satisfies (18). Also, each vector field from $\{\alpha_2^v, \dots, \alpha_n^v, e_1^c, \dots, e_n^c\}$ satisfies (18) at the point $(x, w) \in H_t$. Therefore, (iv) follows, since $\{\alpha_2^v, \dots, \alpha_n^v, e_1^c, \dots, e_n^c\}$ is a linearly independent system of $(n-1)$ vectors tangent to H_t at any point $(x, w) \in H_t$ and complete the proof. $\square$

Remark 8. The vector field ξ^c is not Killing on the whole manifold $(T^*M, \bar{g})$, as $\mathcal{L}_{\xi^c}(E_1, E_1)$ is not everywhere zero. However, if we use the relations (3), (4) and (9) to compute $\mathcal{L}_{\xi^c}\bar{g}$ on the frame of Theorem 7 (iv), we obtain the following:

Corollary 9. *For any $t \in \mathbb{R} - \{0\}$, the vector field ξ^c restricted to H_t is Killing.*

Now, we note that at any point of H_t, the vector fields $\frac{1}{t\sqrt{|b|}}\xi^c, E_k, E_{k*}$, $k=\overline{2,n}$ are tangent to H_t, (as they are all orthogonal to $grad\bar{f}$ with respect to $\bar{g}$). In the above construction (7), we stress that we have the freedom to take $\alpha_2,\ldots,\alpha_n$ anyway, under the only condition that together with α_1 the system $\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ should be linearly independent and hence constitute a basis. Since on H_t, we have $\alpha_1(\xi)=w(\xi)=t\neq 0$, we may assume that $\frac{1}{t}\xi$ is dual to α_1 (and hence ξ is collinear with e_1), which means in the above construction $\alpha_2,\ldots,\alpha_n$ are taken such that $\alpha_k(\xi)=0$. It follows that $\{e_1=\frac{1}{t}\xi,e_2,\ldots,e_n\}$ is the dual basis of $\alpha_1=w,\alpha_2,\ldots,\alpha_n$. One can check that restricted to H_t, the local system $\{\frac{1}{t\sqrt{|b|}}\xi^c, E_k, E_{k*}/k=\overline{2,n}\}$ is orthogonal and from (4), this local system is orthonormal on H_t. With this construction, we have proved the following:

Proposition 10. *For any $t\in\mathbb{R}-\{0\}$, it follows that $\{\frac{1}{t\sqrt{|b|}}\xi^c, E_k, E_{k*}/k=\overline{2,n}\}$, is a local orthonormal frame of the hypersurface H_t.*

Let Ric denote the Ricci tensor field of the natural Riemann extension $\bar{g}$ on T^*M, and let S denote the Ricci tensor field of g_t (the restriction of $\bar{g}$ to H_t). Then, we have the following:

Lemma 11. *At any point $(x,w)\in H_t$, the Ricci tensor field Ric on T^*M is related by the Ricci tensor field S on H_t, by:*

$$\begin{aligned} Ric(A,B) &= S(A,B)-\frac{1}{bt^2}\bar{g}(\bar{R}(\xi^c,A)B,\xi^c)+\frac{c}{tb}\bar{g}(\bar{R}(W,A)B,\xi^c)\\ &\quad +\frac{c}{tb}\bar{g}(\bar{R}(\xi^c,A)B,W)-\frac{c^2}{b}\bar{g}(\bar{R}(W,A)B,W) \end{aligned} \tag{20}$$

for any A,B tangent to H_t in (x,w).

Proof. At any point $(x,w)\in H_t$ the orthonormal system $\{\frac{1}{t\sqrt{|b|}}\xi^c, E_k, E_{k*}\ /\ k =\overline{2,n}\}$ is tangent to H_t, while the vector $N=\frac{a}{t\sqrt{|b|}}grad\bar{f}$ is normal to H_t. Moreover, from (19) and (4), one has $\bar{g}(N,N)=\frac{a^2}{|b|t^2}\bar{g}(grad\bar{f},grad\bar{f})=-\frac{b}{|b|}$. Therefore, $\{\frac{1}{t\sqrt{|b|}}\xi^c, E_k, E_{k*}, N\ /\ k=\overline{2,n}\}$ is an orthonormal basis at (x,w) of T^*M. Hence, at any point $(x,w)\in H_t$, under the above notations we have:

$$\begin{aligned} Ric(A,B) &= S(A,B)-\frac{b}{|b|}\bar{g}(\bar{R}(N,A)B,N)\\ &= S(A,B)-\frac{a^2}{bt^2}\bar{g}(\bar{R}(grad\bar{f},A)B,grad\bar{f}). \end{aligned} \tag{21}$$

By using (19), we obtain (20) and we complete the proof. □

Theorem 12. *Let M^n be a manifold with a symmetric linear connection ∇ whose Ricci tensor is skew-symmetric and let $\bar{g}$ be the proper natural Riemann extension on T^*M. For any $t\in\mathbb{R}-\{0\}$, the hypersurface H_t endowed with the metric g_t, (inherited from $\bar{g}$) is Einstein if and only if*

$$7b=8a. \tag{22}$$

Proof. From Theorem 7 (iv), we need to compute Ric and S for any pair of vectors A, B at $(x, w) \in H_t$, where A, B are arbitrary taken from the basis $\{\alpha_2^v, \dots, \alpha_n^v, e_1^c, \dots, e_n^c\}$. We use (10), (11), (12), (13), (14) and (15), in the following instances:

Case 1: $A = e_k^c$, $B = e_j^c$, $k, j = \overline{1, n}$. By a long calculation, we obtain

$$Ric(e_k^c, e_j^c) = S(e_k^c, e_j^c) + \frac{c}{a}\bar{g}(e_k^c, e_j^c), \quad k, j = \overline{1, n}.$$

From [11], we recall that under the hypothesis of theorem, we have

$$Ric(e_k^c, e_j^c) = \frac{1}{2}\frac{4a + (n-1)b}{a^2}\bar{g}(e_k^c, e_j^c), \quad k, j = \overline{1, n}.$$

Case 2: $A = \alpha_k^v$, $B = e_j^c$, $k = \overline{2, n}$, $j = \overline{1, n}$. A long calculation gives

$$Ric(\alpha_k^v, e_j^c) = S(\alpha_k^v, e_j^c) + \frac{c}{4a}\bar{g}(\alpha_k^v, e_j^c).$$

We recall from [11] that the above hypothesis yields

$$Ric(\alpha_k^v, e_j^c) = \frac{1}{2}\frac{(n+1)b}{a^2}\bar{g}(\alpha_k^v, e_j^c).$$

Case 3: $A = \alpha_k^v$, $B = \alpha_j^v$, $k, j = \overline{2, n}$. We obtain

$$Ric(\alpha_k^v, \alpha_j^v) = S(\alpha_k^v, \alpha_j^v).$$

On the other side, one has:

$$Ric(\alpha_k^v, \alpha_j^v) = 0 = \bar{g}(\alpha_k^v, \alpha_j^v).$$

Hence, from the above three cases, we conclude

$$\begin{aligned} S(e_k^c, e_j^c) &= \frac{1}{2}\frac{4a + (n-1)b - 2ac}{a^2}\bar{g}(e_k^c, e_j^c), \quad k, j = \overline{1, n}, \\ S(\alpha_k^v, e_j^c) &= \frac{(2n+1)b}{4a^2}\bar{g}(\alpha_k^v, e_j^c), \quad k = \overline{2, n}, \quad j = \overline{1, n}, \\ S(\alpha_k^v, \alpha_j^v) &= 0, \quad k, j = \overline{2, n}. \end{aligned}$$

Hence, the restriction of $\bar{g}$ to H_t is Einstein if and only if (22) is satisfied, which complete the proof. ☐

Remark 13. (i) The family of Einstein hypersurfaces are obtained in the Theorem 12 under the necessary and sufficient condition (22), which depends only on the coefficients a and b of the natural Riemann extension, but it is independent on the parameter $t \in \mathbb{R} - \{0\}$. Hence, the condition (22) is the same for the whole family and does not depend on the hypersurface chosen.

(ii) If M is a surface, then any hypersurface H_t, $t \in \mathbb{R} - \{0\}$, is a three-dimensional Lorentzian manifold with equal Ricci eigenvalues. For distinct Ricci eigenvalues, we cite [12].

(iii) A deformed Riemann extension is a semi-Riemannian metric which generalizes the natural Riemann extension, since in (4) the term $bw(X_x)$ $w(Y_x)$ is replaced by $\Phi(X_x, Y_x)$, for any $x \in M$, where Φ is an arbitrary symmetric (0,2)-tensor field, (see [3]). A further study would establish the conditions under which Theorem 12 can be generalized if the natural Riemann extension is replaced with the deformed Riemann extension.

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References

[1] Bejan, C.-L., Kowalski, O.: On some differential operators on natural Riemann extensions. Ann. Glob. Anal. Geom. **48**, 171–180 (2015)

[2] Besse, A.: Einstein manifolds. Springer, Berlin (2008)

[3] Brozos-Vazquez, M., Garcia-Rio, E.: Four dimensional neutral signature self-dual gradient Ricci solitons. Indiana Univ. Math. J. **65**, 1921–1943 (2016)

[4] Burdujan, I.: On almost Cliffordian manifolds. Ital. J. Pure Appl. Math. **13**, 129–144 (2003)

[5] Burdujan, I.: Clifford-Kähler manifolds. Balkan J. Geom. Appl. **13**(2), 12–23 (2008)

[6] Crasmareanu, M., Piscoran, L.-I.: Weak para-CR structures on vertical bundles. Adv. Appl. Clifford Algebras **26**(4), 1127–1136 (2016)

[7] Dunajski, M., Tod, P.: Four dimensional metrics conformal to Kähler. Math. Proc. Camb. Phil. Soc. **148**, 485–504 (2010)

[8] Dunajski, M., Mettler, T.: Gauge theory on projective surfaces and anti-self-dual Einstein metrics in dimension four. arXiv:1509.04276v2

[9] Girard, P.R.: Einsten's equations and Clifford algebra. Adv. Appl. Clifford Algebras **9**(2), 225–230 (1999)

[10] Hall, G., Kırık, B.: Recurrence structures in 4-dimensional manifolds with metric of signature (+,+,-,-). J. Geom. Phys. **98**, 262–274 (2015)

[11] Kowalski, O., Sekizawa, M.: On natural Riemann extensions. Publ. Math. Debrecen **78**, 709–721 (2011)

[12] Kowalski, O., Sekizawa, M.: Existence and classification of three-dimensional Lorentzian manifolds with prescribed distinct Ricci eigenvalues. J. Geom. Phys. **99**, 232–238 (2016)

[13] Moroianu, A., Semmelmann, S.: Clifford structures on Riemannian manifolds. Adv. Math. **228**, 940–967 (2011)

[14] Patterson, E.M., Walker, A.G.: Riemannian extensions. Q. J. Math. Oxford Ser. **2**(3), 19–28 (1952)

[15] Sekizawa, M.: Natural transformations of affine connections on manifolds to metrics on cotangent bundles. In: Proceedings of 14th Winter School on Abstract Analysis (Srni, 1986), Rend. Circ. Mat. Palermo, **14**, 129–142 (1987)

[16] Willmore, T.J.: An introduction to Differential Geometry. Clarendon Press, Oxford (1959)

[17] Yano, K., Paterson, E.M.: Vertical and complete lifts from a manifold to its cotangent bundle. J. Math. Soc. Jpn. **19**, 91–113 (1967)

[18] Yano, K., Ishihara, S.: Tangent and cotangent bundles. Marcel Dekker, New York (1973)

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