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SOME PROPERTIES OF THE p-ADIC LOG BETA FUNCTION

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Abstract. In the present work we consider the p-adic log beta function. By using properties of the Diomand's log gamma function we obtain some results for the p-adic logarithm beta function.

1. Introduction

Let p be an odd prime number. By $\mathbb{Z}_p, \mathbb{Q}_p$ and $\mathbb{C}_p$ we denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. In 1975 Y. Morita [11] defined the p-adic gamma function Γ_p by the formula

$$\Gamma_p(x) = \lim_{n \to x} (-1)^n \prod_{\substack{j<n \\ (p,j)=1}} j$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. The p-adic gamma function $\Gamma_p(x)$ has a great interest and has been studied by D. Barsky (1977) [2], B. Dwork (1983) [5], T. Kim (1997) [7] and others. Diamond defined and studied the p-adic log gamma function $G_p(x)$ (1977) [4]. In 1980 the p-adic beta function is used in Dwork cohomology and an cohomological interpretation of p-adic beta function is given by M. Boyarsky [3]. In 2015, elementary properties of the p-adic beta function was studied [10].

Its is well known that the classical beta function B is defined by the formula

$$B(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1.1}$$

where Γ is the gamma function.

For any function $f \in C^1(\mathbb{Z}_p \to \mathbb{C}_p)$, the Volkenborn integral of f on $\mathbb{Z}_p$ is defined by the formula (see [14])

$$\int_{\mathbb{Z}_p} f(x)dx := \lim_{n \to \infty} p^{-n} \sum_{j=0}^{p^n-1} f(j) \tag{1.2}$$

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We note that the p-adic Bernoulli numbers B_n are defined by the Volkenborn integral

$$\int_{\mathbb{Z}_p} x^n dx = B_n \quad (n \geq 0) \tag{1.3}$$

For example, the first few p-adic Bernoulli numbers are

$$B_0 = \int_{\mathbb{Z}_p} dx = 1, B_1 = \int_{\mathbb{Z}_p} x dx = -\frac{1}{2}, B_2 = \int_{\mathbb{Z}_p} x^2 dx = \frac{1}{6}, B_3 = \int_{\mathbb{Z}_p} x^3 dx = 0. \tag{1.4}$$

We note that q-analogue of the Volkenborn integral is defined by T. Kim [9] and some properties related Bernoulli numbers is given in [12]. We recall that the p-adic logarithm functions is defined by formula

$$\log_p(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n. \tag{1.5}$$

where $x \in \mathbb{C}_p, |x|_p < 1$.

Using definition of Volkenborn integral, J. Diamond (1977) ([4]) gave a definition for the p-adic log gamma function as follows

$$G_p(x) := \int_{\mathbb{Z}_p} ((x+u)\log_p(x+u) - (x+u))du \tag{1.6}$$

for $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$ where $\log_p$ is the Iwasawa p-adic logarithm (see [6]). The p-adic log gamma function and its q-analogue have been in studied in [1] and [8].

The p-adic Stirling series for Diomand's log gamma function is

$$G_p(x) = (x - \frac{1}{2})\log_p x - x + \sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)x^k} \tag{1.7}$$

for $x \in \mathbb{C}_p, |x|_p > 1$, where B_k is the kth p-adic Bernoulli number.

The function G_p has following basic properties (for detail see [4], [13])

(1) For $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$,

$$G_p(x+1) - G_p(x) = \log_p x. \tag{1.8}$$

(2) For $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$,

$$G_p(x) + G_p(1-x) = 0. \tag{1.9}$$

(3) For each $m \in \mathbb{N}$

$$G_p(x) = (x - \frac{1}{2})\log_p m + \sum_{j=0}^{m-1} G_p(\frac{x+j}{m}) \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p) \tag{1.10}$$

In particular, for $n \in \mathbb{N}$

$$G_p(x) = \sum_{j=0}^{p^n-1} G_p(\frac{x+j}{p^n}) \qquad (x \in \mathbb{C}_p \setminus \mathbb{Z}_p) \tag{1.11}$$

(4) Connection with Morita's function Γ_p

$$\log_p \Gamma_p = \sum_{\substack{j=0 \\ |x+j|_p = 1}}^{p-1} G_p(\frac{x+j}{p}) \qquad (x \in \mathbb{Z}_p) \tag{1.12}$$

2. Main Results

Let $E = \{(x, y) : x, y, x + y \in \mathbb{C}_p \backslash \mathbb{Z}_p\}$. We define the p-adic log beta function on E as follows

Definition 1. *We define the p-adic log beta function $\log_p B(x, y)$ by formula*

$$\log_p B(x, y) := G_p(x) + G_p(y) - G_p(x + y). \tag{2.1}$$

for $(x, y) \in E$.

According the properties the Diamond's p-adic log gamma function we obtain the following results, which is properties of $\log_p B(x, y)$.

Theorem 2.1. *If $(x, y) \in E$ then*

$$\begin{aligned}\log_p B(x, y) &= \sum_{n=1}^{\infty} \frac{B_{n+1}}{n(n+1)} \left(\frac{1}{x^n} + \frac{1}{y^n} - \frac{1}{(x+y)^n} \right) + x \log_p x + y \log_p y \\ &\quad -(x + y) \log_p (x + y) - \frac{1}{2} \log_p \frac{xy}{x + y}.\end{aligned}$$

In particular, if n is even and isn't 0 then

$$\log_p B(x, y) = x \log_p x + y \log_p y - (x + y) \log_p (x + y) - \frac{1}{2} \log_p \frac{xy}{x + y}.$$

Proof. Let $(x, y) \in E$. From Definition 1 we know that

$$\log_p B(x, y) = G_p(x) + G_p(y) - G_p(x + y)$$

By (1.7) we see that

$$\begin{aligned}\log_p B(x, y) &= (x - \frac{1}{2}) \log_p x - x + \sum_{n=1}^{\infty} \frac{B_{n+1}}{n(n+1)x^n} + (y - \frac{1}{2}) \log_p y - y + \sum_{n=1}^{\infty} \frac{B_{n+1}}{n(n+1)y^n} \\ &\quad -(x + y - \frac{1}{2}) \log_p (x + y) - (x + y) + \sum_{n=1}^{\infty} \frac{B_{n+1}}{n(n+1)(x+y)^n}\end{aligned}$$

By simple some calculations, we complete the proof. Moreover, using (1.3) and (1.4) we known that if n is even then $B_{n+1} = 0$. Thus, ıt is completed proof of the theorem. □

Theorem 2.2. *If $(x, y) \in E$ then*

$$\log_p B(x + 1, y) - \log_p B(x, y + 1) = \log_p \frac{x}{y}.$$

Proof. Let $(x, y) \in E$. From Definition 1 we have that

$$\log_p B(x + 1, y) = G_p(x + 1) + G_p(y) - G_p(x + 1 + y) \tag{2.2}$$

and

$$\log_p B(x, y + 1) = G_p(x) + G_p(y + 1) - G_p(x + y + 1). \tag{2.3}$$

From (2.2) and (2.3) we get

$$\log_p B(x+1, y) - \log_p B(x, y+1) = G_p(x+1) + G_p(y) - G_p(x) - G_p(y+1) \tag{2.4}$$

Using (1.8) in (2.4) we write

$$\log_p B(x + 1, y) - \log_p B(x, y + 1) = \log_p x - \log_p y.$$

□

Theorem 2.3. *For any $(x, y) \in E$, the relation*

$$\log_p B(x+1, y+1) = \log_p B(x, y) + \log_p \frac{xy}{(x+y+1)(x+y)} \tag{2.5}$$

holds.

Proof. By Definition (1) we obtain that

$$\log_p B(x+1, y+1) = G_p(x+1) + G_p(y+1) - G_p(x+1+y+1) \tag{2.6}$$

Using (1.8) in the equality (2.6) we see that

$$\log_p B(x+1, y+1) = G_p(x) + \log_p x + G_p(y) + \log_p y - G_p(x+1+y) - \log_p(x+y+1)$$

Again, using (1.8) we obtain that

$$\begin{aligned}\log_p B(x+1, y+1) &= G_p(x) + \log_p x + G_p(y) + \log_p y - G_p(x+y) - \log_p(x+y) \\ &\quad - \log_p(x+y+1).\end{aligned}$$

In a similar way, if we rearrange then we obtain the equality (2.5). □

Theorem 2.4. *For any $(x, y) \in E$, the equality*

$$\log_p B(x+1, y+1) + \log_p B(-x, -y) = -\log_p(x+y+1).$$

holds.

Proof. Let $(x, y) \in E$. Using Definition (1) we know that

$$\log_p B(x+1, y+1) = G_p(x+1) + G_p(y+1) - G_p(x+1+y+1) \tag{2.7}$$

and

$$\log_p B(-x, -y) = G_p(-x) + G_p(-y) - G_p(-x-y) \tag{2.8}$$

From (2.7) and (2.8) we get that

$$\begin{aligned}\log_p B(x+1, y+1) + \log_p B(-x, -y) &= G_p(x+1) + G_p(y+1) - G_p(x+y+2) + \\ &\quad G_p(-x) + G_p(-y) - G_p(-x-y) \qquad (2.9)\end{aligned}$$

Using $-x$ replace x in equality (1.9) we can write

$$G_p(-x) + G_p(1+x) = 0. \tag{2.10}$$

Then, from (2.9) and (2.10) we have that

$$\log_p B(x+1, y+1) + \log_p B(-x, -y) = -G_p(x+1+y+1) - G_p(-x-y).$$

Using (1.8) we see that

$$\log_p B(x+1, y+1) + \log_p B(-x, -y) = -G_p(1+(x+y)) - \log_p(x+y+1) - G_p(-(x+y))$$

By equality (2.10) the proof is completed. □

Theorem 2.5. *If $x, y \in E$ and $m \in \mathbb{N}^+$, then*

$$\begin{aligned}\log_p B(x, y) &= \sum_{j=0}^{m-1} \log_p B\left(\frac{x+j}{m}, \frac{y+j}{m}\right) + \sum_{j=\frac{m}{2}}^{\frac{2m-3}{2}} G_p\left(\frac{x+y+2j+1}{m}\right) - \\ &\quad - \sum_{j=0}^{\frac{m-3}{2}} G_p\left(\frac{x+y+2j+1}{m}\right) - \frac{1}{2}\log_p m\end{aligned}$$

In particular, for each $n \in \mathbb{N}$

$$\begin{aligned}\log_p B(x,y) &= \sum_{j=0}^{p^n-1} \log_p B(\frac{x+j}{p^n}, \frac{y+j}{p^n}) + \sum_{j=0}^{\frac{p^n-3}{2}} G_p(\frac{x+y+p^n+2j+1}{p^n}) \\ &\quad - \sum_{j=0}^{\frac{p^n-3}{2}} G_p(\frac{x+y+2j+1}{p^n}).\end{aligned}$$

Proof. Let $x, y \in \mathbb{C}_p \backslash \mathbb{Z}_p$ and $m \in \mathbb{N}^+$. From Definition 1 and (1.10) we know that

$$\log_p B(x,y) = G_p(x) + G_p(y) - G_p(x+y)$$

$$\begin{aligned}\log_p B(x,y) &= (x - \frac{1}{2}) \log_p m + \sum_{j=0}^{m-1} G_p(\frac{x+j}{m}) + (y - \frac{1}{2}) \log_p m + \sum_{j=0}^{m-1} G_p(\frac{y+j}{m}) \\ &\quad -(x+y-\frac{1}{2}) \log_p m - \sum_{j=0}^{m-1} G_p(\frac{x+y+j}{m})\end{aligned}$$

or

$$\log_p B(x,y) = \sum_{j=0}^{m-1} G_p(\frac{x+j}{m}) + \sum_{j=0}^{m-1} G_p(\frac{y+j}{m}) - \sum_{j=0}^{m-1} G_p(\frac{x+y+j}{m}) - \frac{1}{2} \log_p m$$

Hence we reach the desired result

$$\begin{aligned}\log_p B(x,y) &= \sum_{j=0}^{m-1} \log_p B(\frac{x+j}{m}, \frac{y+j}{m}) + \sum_{j=0}^{\frac{m-3}{2}} G_p(\frac{x+y+m+2j+1}{m}) \\ &\quad - \sum_{j=0}^{\frac{m-3}{2}} G_p(\frac{x+y+2j+1}{m}) - \frac{1}{2} \log_p m\end{aligned}$$

In particular, if $m = p^n$ then from (1.11) we have that

$$\begin{aligned}\log_p B(x,y) &= \sum_{j=0}^{p^n-1} \log_p B(\frac{x+j}{p^n}, \frac{y+j}{p^n}) + \sum_{j=0}^{\frac{p^n-3}{2}} G_p(\frac{x+y+p^n+2j+1}{p^n}) \\ &\quad - \sum_{j=0}^{\frac{p^n-3}{2}} G_p(\frac{x+y+2j+1}{p^n})\end{aligned}$$

□

In what following connection between $\log_p B(x,y)$ and $\log_p B_p(x,y)$:

Theorem 2.6. *For $|x|_p < 1$, $|y|_p < 1$ and $|x+y|_p < 1$, the equality holds:*

$$\begin{aligned}\log_p B_p(x,y) &= \sum_{j=1}^{p-1} \log_p B(\frac{x+j}{p}, \frac{y+j}{p}) - \sum_{j=0}^{\frac{p-3}{2}} G_p(\frac{x+y+2j+1}{p}) \\ &\quad + \sum_{j=0}^{\frac{p-3}{2}} G_p(\frac{x+y+p+2j+1}{p})\end{aligned}$$

Proof. Assume that $|x|_p < 1$, $|y|_p < 1$, $|x+y|_p < 1$. We know that

$$\log_p B_p(x,y) = \log_p \Gamma_p(x) + \log_p \Gamma_p(y) - \log_p \Gamma_p(x+y)$$

and also

$$\log_p \Gamma_p(a) = \sum_{\substack{j=0 \\ |a+j|_p=1}}^{p-1} G_p(\frac{a+j}{p}) \qquad (a \in \mathbb{Z}_p).$$

Since $|x|_p < 1$, $|y|_p < 1$, $|x+y|_p < 1$, then $|x+j|_p = 1, |y+j|_p = 1$ and $|x+y+j|_p = 1$ for all $1 \le j \le p-1$. Hence, we can write

$$\log_p B_p(x,y) = \sum_{j=1}^{p-1} G_p(\frac{x+j}{p}) + \sum_{j=1}^{p-1} G_p(\frac{y+j}{p}) - \sum_{j=1}^{p-1} G_p(\frac{x+y+j}{p}).$$

or

$$\begin{aligned} \log_p B_p(x,y) &= G_p(\frac{x+1}{p}) + G_p(\frac{y+1}{p}) - G_p(\frac{x+y}{p}) + ... + G_p(\frac{x+\frac{p-1}{2}}{p}) \\ &+ G_p(\frac{y+\frac{p-1}{2}}{p}) - G_p(\frac{x+y+p-1}{p}) + G_p(\frac{x+\frac{p+1}{2}}{p}) + ... + \\ &+ G_p(\frac{x+p-1}{p}) + G_p(\frac{y+\frac{p+1}{2}}{p}) + ... + G_p(\frac{y+p-1}{p}) - \\ &- G_p(\frac{x+y+1}{p}) - G_p(\frac{x+y+3}{p}) - ... - G_p(\frac{x+y+p-2}{p}) \end{aligned}$$

Using Definition 1 and the equality above we obtain

$$\begin{aligned} \log_p B_p(x,y) &= \sum_{j=0}^{\frac{p-1}{2}} \log_p B(\frac{x+j}{p}, \frac{y+j}{p}) + \sum_{j=\frac{p+1}{2}}^{p-1} G_p(\frac{x+j}{p}) + \sum_{j=\frac{p+1}{2}}^{p-1} G_p(\frac{y+j}{p}) - \\ &- \sum_{j=0}^{\frac{p-3}{2}} G_p(\frac{x+y+2j+1}{p}) \end{aligned}$$

By simple calculation and Definition 1 we can have that

$$\begin{aligned} \log_p B_p(x,y) &= \sum_{j=1}^{\frac{p-1}{2}} \log_p B(\frac{x+j}{p}, \frac{y+j}{p}) + \sum_{j=\frac{p+1}{2}}^{p-1} \log_p B(\frac{x+j}{p}, \frac{y+j}{p}) + \\ &+ \sum_{j=0}^{\frac{p-3}{2}} G_p(\frac{x+y+p+2j+1}{p}) - \sum_{j=0}^{\frac{p-3}{2}} G_p(\frac{x+y+2j+1}{p}). \end{aligned}$$

This completes the proof. □

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