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SOME PROPERTIES OF TH q-EXTENSION OF THE p-ADIC BETA FUNCTION

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ABSTRACT. In the present work we study the q-extension of p-adic analogue of the classical beta function. We obtain some properties of the q-extension of the p-adic beta function.

Key Words: p-adic number, p-adic beta function, q-extension of the p-adic beta function.

1. PRELIMINARIES

In this paper, let p be a prime number and $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. In 1975 Y. Morita [19] defined the p-adic gamma function Γ_p by the formula

$$\Gamma_p(x) = \lim_{n\to x}(-1)^n \prod_{\substack{j<n \\ (p,j)=1}} j$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. The p-adic gamma function $\Gamma_p(x)$ has a great interest and has been studied by J. Diamond (1977) [6], D. Barsky (1977) [3], B. Dwork (1983) [7], T. Kim (1997) [13] and others. B. Gross and N. Koblitz (1979) [8], H. Cohen and E. Friedman (2008) [5] and I. Shapiro (2012) [21] studied the relationship between some special functions and the p-adic gamma function $\Gamma_p(x)$.

The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ is defined by N. Koblitz [10] as follows: Let $q \in \mathbb{C}_p, |q-1|_p < 1, q \neq 1$. The q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$ is defined by formula

$$\Gamma_{p,q}(x) = \lim_{n\to x}(-1)^n \prod_{\substack{j<n \\ (p,j)=1}} \frac{1-q^j}{1-q}.$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. We recall that $\lim_{q\to 1}\Gamma_{p,q} = \Gamma_p$.

N. Koblitz (1980, 1982) [10], [11], H. Nakazato (1988) [12], Y. S. Kim (1998) [15] and others investigated the q-extension of the p-adic gamma function $\Gamma_{p,q}(x)$. In [18] (2013) some properties of the q-extension of the p-adic gamma function were given.

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A p-adic analogue of classical beta function can be defined by the formula

$$B_p(x,y) := \frac{\Gamma_p(x)\,\Gamma_p(y)}{\Gamma_p(x+y)}, \quad x,y \in \mathbb{Z}_p$$

In 1980 M. Boyarsky [4] used to the p-adic beta function in Dwork cohomology and gave an cohomological interpretation of the p-adic beta function. In 2006 F. Baldassarri [2] considered two constructions of the p-adic beta functions as the p-adic etale and p-adic crystalline beta functions. In [16] the basic properties of the p-adic beta function were given. In [17] the Gauss-Legendre multiplication type formulas were derived for the p-adic beta function. Also, the q-extensions of some special functions in p-adic analysis have been studied by many authors (see, [1], [9], [14], [20] and others).

In the present work we define the q-extension of the p-adic beta $B_{p,q}$ and we obtain some properties of $B_{p,q}$. To prove our results we use the following properties of the q-extension of the p-adic gamma $\Gamma_{p,q}$.

Lemma 1.1. *$\Gamma_{p,q}$ has the following properties:*

(i) *For all $x \in \mathbb{Z}_p$, $\Gamma_{p,q}(x+1) = h_{p,q}(x)\Gamma_{p,q}(x)$ where*

$$h_{p,q}(x) = \begin{cases} -\frac{1-q^x}{1-q} & \text{if } |x|_p = 1 \\ -1 & \text{if } |x|_p < 1 \end{cases}$$

(ii) *$\Gamma_{p,q}(1) = -1$ and $\Gamma_{p,q}(0) = 1$.*

(iii) *For any p, $\Gamma_{p,q}(-n)$ $(n \in \mathbb{N})$ is given by*

$$\Gamma_{p,q}(-n) = (-1)^{n+1-\left[\frac{n}{p}\right]} \prod_{\substack{j<n+1 \\ (p,j)=1}} q^j (\Gamma_p(n+1))^{-1}. \tag{1.1}$$

(iv) *If $p \neq 2$, then for all $x \in \mathbb{Z}_p$*

$$\Gamma_{p,q}(x)\Gamma_{p,q}(1-x) = (-1)^{\ell(x)} \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} q^j \tag{1.2}$$

and if $p = 2$ then for all $x \in \mathbb{Z}_2$

$$\Gamma_{p,q}(x)\Gamma_{p,q}(1-x) = (-1)^{\sigma_1(x)+1} \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} q^j \tag{1.3}$$

where $\ell : \mathbb{Z}_p \to \{1,2,\ldots,p\}$ assigns to $x \in \mathbb{Z}_p$ its residue $\in \{1,2,\ldots,p\}$ modulo $p\mathbb{Z}_p$ and where σ_1 is defined by the formula

$$\sigma_1\left(\sum_{j=0}^{\infty} a_j 2^j\right) = a_1$$

Corollary 1.2 ([15]). *Let $p \neq 2$. We get*

$$\Gamma_{p,q}\left(\frac{1}{2}\right)^2 = (-1)^{\ell(\frac{1}{2})} \lim_{n\to\frac{1}{2}} \prod_{\substack{j<n \\ (p,j)=1}} q^j$$

Now $\ell(\frac{1}{2}) = \ell(\frac{1}{2}(p+1)) = \frac{1}{2}(p+1)$ so that

$$\Gamma_{p,q}\left(\frac{1}{2}\right)^2 = \begin{cases} \lim\limits_{n\to\frac{1}{2}} \prod\limits_{\substack{j<n \\ (p,j)=1}} q^j & \text{if } p \equiv 3 \pmod 4 \\ -\lim\limits_{n\to\frac{1}{2}} \prod\limits_{\substack{j<n \\ (p,j)=1}} q^j & \text{if } p \equiv 1 \pmod 4 \end{cases} \tag{1.4}$$

2. THE q-EXTENSION OF THE p-ADIC BETA FUNCTION

Using the q-extension of the p-adic gamma function, we can define q-extension of the p-adic beta function $B_{p,q}$

Definition 2.1. Let $q \in \mathbb{C}_p$, $|q-1|_p < 1$, $q \neq 1$. The q-extension of the p-adic beta function $B_{p,q}$ is defined by

$$B_{p,q}(x,y) := \frac{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)}.$$

for $x, y \in \mathbb{Z}_p$. We note that $\lim_{q\to 1} B_{p,q} = B_p$.

Now, we give some properties of the q-extension of the p-adic beta function.

Theorem 2.2. *The q-extension of the p-adic beta function is symmetric about x and y. Namely,*

$$B_{p,q}(x,y) = B_{p,q}(y,x)$$

for $x, y \in \mathbb{Z}_p$.

Proof. It follows immediately from Definition 2.1 for $x, y \in \mathbb{Z}_p$. □

Theorem 2.3. *If $p \neq 2$ then the equality holds:*

$$B_{p,q}(x,y)B_{p,q}(x+y,1-y) = \frac{(-1)^{\ell(y)}}{h_{p,q}(x)} \lim_{n\to y} \prod_{\substack{j<n \\ (p,j)=1}} q^j$$

and if $p = 2$ then the equality holds:

$$B_{p,q}(x,y)B_{p,q}(x+y,1-y) = \frac{(-1)^{\sigma_1(y)+1}}{h_{p,q}(x)} \lim_{n\to y} \prod_{\substack{j<n \\ (p,j)=1}} q^j$$

where

$$h_{p,q}(x) = \begin{cases} -\frac{1-q^x}{1-q} & \text{if } |x|_p = 1 \\ -1 & \text{if } |x|_p < 1 \end{cases}$$

and $\ell : \mathbb{Z}_p \to \{1, 2, \ldots, p\}$ *assigns to* $y \in \mathbb{Z}_p$ *its residue* $\in \{1, 2, \ldots, p\}$ *modulo* $p\mathbb{Z}_p$ *holds for* $x, y \in \mathbb{Z}_p$ *and where* σ_1 *is defined by the formula* $\sigma_1(\sum_{j=0}^{\infty} a_j 2^j) = a_1$.

Proof. Let $x, y \in \mathbb{Z}_p$. From Definition 2.1 we know that

$$B_{p,q}(x, y)B_{p,q}(x + y, 1 - y) = \frac{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}(x + y)} \frac{\Gamma_{p,q}(x + y)\Gamma_{p,q}(1 - y)}{\Gamma_{p,q}(x + 1)}.$$

Then, by using Lemma 1.1(i) we get

$$B_{p,q}(x, y)B_{p,q}(x + y, 1 - y) = \frac{\Gamma_{p,q}(x)\Gamma_{p,q}(y)\Gamma_{p,q}(1 - y)}{\Gamma_{p,q}(x)h_{p,q}(x)}. \tag{2.1}$$

Assume that $p \neq 2$. Then, from (1.2) in Lemma 1.1 (iv) we obtain the formula

$$B_{p,q}(x, y)B_{p,q}(x + y, 1 - y) = \frac{(-1)^{\ell(y)}}{h_{p,q}(x)} \lim_{n \to y} \prod_{\substack{j<n \\ (p,j)=1}} q^j.$$

Let $p = 2$. Using (1.3) in (2.1) the proof is completed. □

Theorem 2.4. *The equality*

$$B_{p,q}(x + 1, y) = \frac{h_{p,q}(x)}{h_{p,q}(x + y)} B_{p,q}(x, y) \tag{2.2}$$

holds for all $x, y \in \mathbb{Z}_p$.

Proof. Let $x, y \in \mathbb{Z}_p$. By using Definition 2.1 and Lemma 1.1 (i) we have

$$\begin{aligned}
B_{p,q}(x + 1, y) &= \frac{\Gamma_{p,q}\,(x + 1)\,\Gamma_{p,q}(y)}{\Gamma_{p,q}(x + 1 + y)} \\
&= \frac{\Gamma_{p,q}\,(x)\,h_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}((x + y) + 1)} \\
&= \frac{\Gamma_{p,q}\,(x)\,h_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}(x + y)h_{p,q}(x + y)} \\
&= \frac{h_{p,q}(x)}{h_{p,q}(x + y)} \frac{\Gamma_{p,q}\,(x)\,\Gamma_{p,q}(y)}{\Gamma_{p,q}(x + y)} \\
&= \frac{h_{p,q}(x)}{h_{p,q}(x + y)} B_{p,q}(x, y).
\end{aligned}$$

□

Theorem 2.5. *The following equality holds:*

$$B_{p,q}(x, y + 1) = \frac{h_{p,q}(y)}{h_{p,q}(x + y)} B_{p,q}(x, y) \tag{2.3}$$

for all $x, y \in \mathbb{Z}_p$.

Proof. From Theorem 2.2 and Theorem 2.4 the theorem is proved. □

Corollary 2.6. *For all $x, y \in \mathbb{Z}_p$*

$$B_{p,q}(x+1,y)+B_{p,q}(x,y+1)=\frac{h_{p,q}(x)+h_{p,q}(y)}{h_{p,q}(x+y)}B_{p,q}(x,y).$$

Proof. Combining (2.2) and (2.3) this corollary is obtained. □

Corollary 2.7. *For all $x, y \in \mathbb{Z}_p$,*

$$B_{p,q}(x,y+1)=\frac{h_{p,q}(y)}{h_{p,q}(x)}B_{p,q}(x+1,y).$$

Proof. Let $x, y \in \mathbb{Z}_p$. From Theorem 2.4 we have

$$B_{p,q}(x,y)=\frac{h_{p,q}(x+y)}{h_{p,q}(x)}B_{p,q}(x+1,y). \tag{2.4}$$

Using (2.4) in Theorem 2.5 we obtain

$$\begin{aligned}B_{p,q}(x,y+1)&=\frac{h_{p,q}(y)}{h_{p,q}(x+y)}\frac{h_{p,q}(x+y)}{h_{p,q}(x)}B_{p,q}(x+1,y)\\&=\frac{h_{p,q}(y)}{h_{p,q}(x)}B_{p,q}(x+1,y).\end{aligned}$$

□

Theorem 2.8. *The equality*

$$B_{p,q}(x+1,y+1)=\frac{h_{p,q}(x)h_{p,q}(y)}{h_{p,q}(x+y+1)h_{p,q}(x+y)}B_{p,q}(x,y)$$

holds for all $x, y \in \mathbb{Z}_p$.

Proof. For $x, y \in \mathbb{Z}_p$, from Definition 2.1, we can write

$$B_{p,q}(x+1,y+1)=\frac{\Gamma_{p,q}(x+1)\,\Gamma_{p,q}(y+1)}{\Gamma_{p,q}(x+1+y+1)}.$$

Using Lemma 1.1(i) we get

$$\begin{aligned}B_{p,q}(x+1,y+1)&=\frac{\Gamma_{p,q}(x+1)\Gamma_{p,q}(y)h_{p,q}(y)}{\Gamma_{p,q}(x+y+1)h_{p,q}(x+y+1)}\\&=\frac{h_{p,q}(y)}{h_{p,q}(x+y+1)}B_{p,q}(x+1,y).\end{aligned} \tag{2.5}$$

Using Theorem 2.4 in (2.5) we prove the theorem. □

Corollary 2.9. *For all $x, y, z \in \mathbb{Z}_p$*

$$B_{p,q}(x,y)B_{p,q}(x+y,z)B_{p,q}(x+y+z,w)=\frac{\Gamma_{p,q}(x)\,\Gamma_{p,q}(y)\,\Gamma_{p,q}(z)\,\Gamma_{p,q}(w)}{\Gamma_{p,q}(x+y+z+w)}.$$

Proof. The corollary is easily proved with a little rearranging and Definition 2.1. □

Theorem 2.10. *The following equalities holds:*
(i) If $p \neq 2$, then

$$B_{p,q}(x,1-x) = (-1)^{\ell(x)+1} \lim \prod_{\substack{j<n \\ (p,j)=1}} q^j$$

(ii) If $p = 2$, then

$$B_{p,q}(x,1-x) = (-1)^{\sigma_1(x)+2} \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} q^j$$

for all $x \in \mathbb{Z}_p$.

Proof. Let $x \in \mathbb{Z}_p$. From Definition 2.1, we have

$$\begin{aligned} B_{p,q}(x,1-x) &= \frac{\Gamma_{p,q}(x)\,\Gamma_{p,q}(1-x)}{\Gamma_{p,q}(x+1-x)} \\ &= \frac{\Gamma_{p,q}(x)\,\Gamma_{p,q}(1-x)}{\Gamma_{p,q}(1)}. \end{aligned}$$

Note that $\Gamma_{p,q}(1) = -1$. Therefore, if $p \neq 2$, then from (1.2)

$$B_{p,q}(x,1-x) = -(-1)^{\ell(x)} \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} q^j = (-1)^{\ell(x)+1} \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} q^j.$$

If $p = 2$, then from (1.3)

$$B_{p,q}(x,1-x) = -(-1)^{\sigma_1(x)+1} \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} q^j = (-1)^{\sigma_1(x)+2} \lim_{n\to x} \prod_{\substack{j<n \\ (p,j)=1}} q^j.$$

□

Corollary 2.11. *If $p \neq 2$, then*

$$B_{p,q}\left(\frac{1}{2},\frac{1}{2}\right) = \begin{cases} -\lim\limits_{n\to\frac{1}{2}} \prod\limits_{\substack{j<n \\ (p,j)=1}} q^j & \text{if } p \equiv 3 \pmod 4 \\ \lim\limits_{n\to\frac{1}{2}} \prod\limits_{\substack{j<n \\ (p,j)=1}} q^j & \text{if } p \equiv 1 \pmod 4 \end{cases} \tag{2.6}$$

Proof. Using Definition 2.1 and Lemma 1.1 (ii), we have

$$B_{p,q}\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma_{p,q}(\frac{1}{2})\Gamma_{p,q}(\frac{1}{2})}{\Gamma_{p,q}(1)} = -\Gamma_{p,q}\left(\frac{1}{2}\right)^2$$

By using (1.4) we obtain equality (2.6) above. □

The beta function can be defined as binomial coefficient indices. The following relation holds for the q-extension of the p-adic beta function.

Theorem 2.12. *The equality*

$$\binom{n}{k}_{p,q} B_{p,q}(n-k+1,k+1) = -\frac{1}{h_{p,q}(n+1)}.$$

holds for all $n,k\in\mathbb{N}$, $k\le n$. Here, the notation $\binom{n}{k}_{p,q}$ is defined by

$$\binom{n}{k}_{p,q} = \frac{(n!)_{p,q}}{((n-k)!)_{p,q}(k!)_{p,q}}$$

and the q-extension of p-adic factorial $(m!)_{p,q}$ is defined by

$$(m!)_{p,q} = \prod_{\substack{1\le j\le m\\(p,j)=1}} \frac{1-q^j}{1-q}$$

for any non negative integer m.

Proof. Assume that $n,k\in\mathbb{N}$, $k\le n$. We know that

$$(n!)_{p,q} = (-1)^{n+1}\Gamma_{p,q}(n+1).$$

Then we can write

$$\begin{aligned}\binom{n}{k}_{p,q} B_{p,q}(n-k+1,k+1) &= \frac{(n!)_{p,q}}{(k!)_{p,q}((n-k)!)_{p,q}} B_{p,q}(n-k+1,k+1)\\ &= \frac{(-1)^{n+1}\Gamma_{p,q}(n+1)}{(-1)^{n-k+1}\Gamma_{p,q}(n-k+1)(-1)^{k+1}\Gamma_{p,q}(k+1)}\times\\ &\quad\times B_{p,q}(n-k+1,k+1)\end{aligned}$$

From Definition 2.1 and by some computations we have

$$\binom{n}{k}_{p,q} B_{p,q}(n-k+1,k+1) = \frac{-\Gamma_{p,q}(n+1)}{\Gamma_{p,q}(n-k+1)\Gamma_{p,q}(k+1)} \frac{\Gamma_{p,q}(n-k+1)\Gamma_{p,q}(k+1)}{\Gamma_{p,q}(n+2)}$$

Using Lemma 1.1(i) we easily see that

$$\begin{aligned}\binom{n}{k}_{p,q} B_{p,q}(n-k+1,k+1) &= -\frac{\Gamma_{p,q}(n+1)}{\Gamma_{p,q}(n+1)h_{p,q}(n+1)}\\ &= -\frac{1}{h_{p,q}(n+1)}.\end{aligned}$$

□

In what follows, we indicate q-extension of p-adic beta function for negative integer numbers

Theorem 2.13. *If $n,m\in\mathbb{N}$, then*

$$B_{p,q}(-n,-m) = (-1)^{1+\left[\frac{n+m}{p}\right]-\left[\frac{n}{p}\right]-\left[\frac{m}{p}\right]} \frac{h_{p,q}(n+m)}{h_{p,q}(n)h_{p,q}(m)} \frac{1}{B_{p,q}(n,m)} \frac{\prod\limits_{\substack{j<m+1\\(p,j)=1}} q^j}{\prod\limits_{\substack{j=n+1\\(p,j)=1}}^{n+m} q^j}$$

where $[j]$ denotes the greatest integer less than or equal to j.

Proof. Let $n, m \in \mathbb{N}$. Using Lemma 1.1 (iii) and Definition 2.1 we get

$$\begin{aligned} B_{p,q}(-n,-m) &= \frac{\Gamma_{p,q}(-n)\Gamma_{p,q}(-m)}{\Gamma_{p,q}(-n-m)} \\ &= \frac{(-1)^{n+1-\left[\frac{n}{p}\right]} \prod\limits_{\substack{j<n+1 \\ (p,j)=1}} q^j (-1)^{m+1-\left[\frac{m}{p}\right]} \prod\limits_{\substack{j<m+1 \\ (p,j)=1}} q^j \Gamma_{p,q}(n+m+1)}{\Gamma_{p,q}(n+1)\Gamma_{p,q}(m+1)(-1)^{n+m+1-\left[\frac{n+m}{p}\right]} \prod\limits_{\substack{j<n+m+1 \\ (p,j)=1}} q^j} \\ &= \frac{(-1)^{1+\left[\frac{n+m}{p}\right]-\left[\frac{n}{p}\right]-\left[\frac{m}{p}\right]} \prod\limits_{\substack{j<m+1 \\ (p,j)=1}} q^j \Gamma_{p,q}(n+m+1)}{\Gamma_{p,q}(n+1)\Gamma_{p,q}(m+1) \prod\limits_{\substack{n<j<n+m+1 \\ (p,j)=1}} q^j} \end{aligned}$$

and by Lemma 1.1(i) we have

$$\begin{aligned} B_{p,q}(-n,-m) &= (-1)^{1+\left[\frac{n+m}{p}\right]-\left[\frac{n}{p}\right]-\left[\frac{m}{p}\right]} \frac{\Gamma_{p,q}(n+m)h_{p,q}(n+m)}{\Gamma_{p,q}(n)h_{p,q}(n)\Gamma_{p,q}(m)h_{p,q}(m)} \frac{\prod\limits_{\substack{j<m+1 \\ (p,j)=1}} q^j}{\prod\limits_{\substack{j=n+1 \\ (p,j)=1}}^{n+m} q^j} \\ &= (-1)^{1+\left[\frac{n+m}{p}\right]-\left[\frac{n}{p}\right]-\left[\frac{m}{p}\right]} \frac{h_{p,q}(n+m)}{h_{p,q}(n)h_{p,q}(m)} \frac{1}{B_{p,q}(n,m)} \frac{\prod\limits_{\substack{j<m+1 \\ (p,j)=1}} q^j}{\prod\limits_{\substack{j=n+1 \\ (p,j)=1}}^{n+m} q^j}. \end{aligned}$$

□

Theorem 2.14. *Let $n, m \in \mathbb{N}$. If $m < n$ then the following equality holds:*

$$B_{p,q}(-n,m) = \prod_{\substack{j=n-m+1 \\ (p,j)=1}}^{n} q^j \frac{h_{p,q}(n-m)(-1)^{m-\left[\frac{n}{p}\right]+\left[\frac{n-m}{p}\right]}}{h_{p,q}(n)} B_{p,q}(n-m,m).$$

If $n \leq m$ then the following equality holds:

$$B_{p,q}(-n,m) = \prod_{\substack{j<n+1 \\ (p,j)=1}} q^j \frac{(-1)^{n+1-\left[\frac{n}{p}\right]}}{h_{p,q}(n)} (B_{p,q}(n,m-n))^{-1}.$$

Proof. Let $n, m \in \mathbb{N}$. From Definition 2.1, we have

$$B_{p,q}(-n,m) = \frac{\Gamma_{p,q}(-n)\Gamma_{p,q}(m)}{\Gamma_{p,q}(-n+m)}.$$

Assume that $n \leq m$. From (1.1) we have

$$B_{p,q}(-n,m) = \frac{(-1)^{n+1-\left[\frac{n}{p}\right]} \prod_{\substack{j<n+1 \\ (p,j)=1}} q^j (\Gamma_p(n+1))^{-1} \Gamma_{p,q}(m)}{\Gamma_{p,q}(-n+m)}.$$

By Lemma 1.1(i) we obtain

$$\begin{aligned} B_{p,q}(-n,m) &= (-1)^{n+1-\left[\frac{n}{p}\right]} \frac{\Gamma_{p,q}(m) \prod_{\substack{j<n+1 \\ (p,j)=1}} q^j}{\Gamma_{p,q}(-n+m)\Gamma_{p,q}(n)h_{p,q}(n)} \\ &= \frac{(-1)^{n+1-\left[\frac{n}{p}\right]} \prod_{\substack{j<n+1 \\ (p,j)=1}} q^j}{h_{p,q}(n)} (B_{p,q}(n,m-n))^{-1}. \end{aligned}$$

Now, let $n > m$. From (1.1) we get

$$\begin{aligned} B_{p,q}(-n,m) &= \frac{(-1)^{n+1-\left[\frac{n}{p}\right]} \prod_{\substack{j<n+1 \\ (p,j)=1}} q^j (\Gamma_{p,q}(n+1))^{-1} \Gamma_{p,q}(m)}{(-1)^{n-m+1-\left[\frac{n-m}{p}\right]} \prod_{\substack{j<n-m+1 \\ (p,j)=1}} q^j (\Gamma_{p,q}(n-m+1))^{-1}} \\ &= (-1)^{m-\left[\frac{n}{p}\right]+\left[\frac{n-m}{p}\right]} \frac{\prod_{\substack{j<n+1 \\ (p,j)=1}} q^j}{\prod_{\substack{j<n-m+1 \\ (p,j)=1}} q^j} \frac{\Gamma_{p,q}(n-m+1)\Gamma_{p,q}(m)}{\Gamma_{p,q}(n+1)}. \end{aligned}$$

Using Lemma 1.1(i) and by some computations, we get

$$\begin{aligned} B_{p,q}(-n,m) &= (-1)^{m-\left[\frac{n}{p}\right]+\left[\frac{n-m}{p}\right]} \prod_{\substack{j=n-m+1 \\ (p,j)=1}}^{n} q^j \frac{h_{p,q}(n-m)\Gamma_{p,q}(n-m)\Gamma_{p,q}(m)}{h_{p,q}(n)\Gamma_{p,q}(n)} \\ &= (-1)^{m-\left[\frac{n}{p}\right]+\left[\frac{n-m}{p}\right]} \prod_{\substack{j=n-m+1 \\ (p,j)=1}}^{n} q^j \frac{h_{p,q}(n-m)}{h_{p,q}(n)} B_{p,q}(n-m,m). \end{aligned}$$

□

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