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On Convergence of Deferred Nörlund and Deferred Riesz Means of Mellin-Fourier Series

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Abstract. In this paper, we examined convergence of the Mellin-Fourier series of recurrent functions by using Deferred Nörlund means $(D_a^b N, p)$ and Deferred Riesz means $(D_a^b R, p)$. Also some important results were obtained.

1 Introduction

The Fourier analysis and Fourier series has been studied for over the century. Many alternatives of the Fourier series have been set for the different type of functions. Mellin-Fourier series is one of them to handle with recurrent functions [3,4,5,7,8,9,10,11,12]. The theory of recurrent functions with a counterpart of the Fourier series in Mellin setting has been investigated in [4,5,6].

Let $f : \mathbb{R}_+ \to \mathbb{C}$. If $f(e^{2\pi}x) = f(x)$, $\forall x \in \mathbb{R}_+$ then f is called recurrent function. If, for $c \in \mathbb{R}$ and $\forall x \in \mathbb{R}_+$, $e^{2\pi c} f(e^{2\pi}x) = f(x)$ then f is called c-recurrent function. As examples, every polynomial in x^i, $\sqrt{-1} = i$, namely $f(x) := \sum_{j=-n}^{n} a_j x^{ij}$, $a_j \in \mathbb{C}$, is a recurrent function and $g(x) := x^{-c} f(x)$ is a c-recurrent function. Y_c function spaces was defined as follows

$$Y_c = \left\{ f \in L^1_{loc}(\mathbb{R}_+) : f \quad \text{is} \quad c-recurrent; ||f||_{Y_c} := \int_{e^{-\pi}}^{e^{\pi}} |f(u)| u^{c-1} du < \infty \right\}, \quad c \in \mathbb{R}.$$

The fundamental interval of c-recurrent functions can be taken as $[e^{-\pi}, e^{\pi}]$ or, more genarally, as $[\lambda e^{-\pi}, \lambda e^{\pi}]$ for any $\lambda > 0$; it is the counterpart of the interval $[-\pi, \pi]$ or $[-\pi + \alpha, \pi + \alpha]$ for any $\alpha \in \mathbb{R}$ in the 2π-periodic case. Observe that the space Y_c, $c \in \mathbb{R}$ is a Banach space with the norm $\|.\|_{Y_c}$. Also, $f(x) := \sum_{j=-n}^{n} a_j x^{ij}$, $a_j \in \mathbb{C}$ belongs to Y_0 and $g(x) := x^{-c} f(x)$ belongs to the Y_c. Mellin-Fourier series of $f \in Y_c$ is defined as

$$f(x) \sim \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \mathcal{M}^c(f;k) x^{-c-ik}, \quad x \in \mathbb{R}_+, \tag{1.1}$$

where $M^c(f;k)$ is the finite Mellin transform of f at $k \in \mathbb{Z}$ defined by

$$M^c(f;k) = \int_{e^{-\pi}}^{e^{\pi}} f(u) u^{c+ik-1} du.$$

As to the fundamental orthogonal system in the Mellin frame, the counterpart of the system $\{e^{ikx}\}_{k\in\mathbb{Z}}$ in the periodic case, we have the functions $\varphi_{c,k}(x) := x^{-(c+ik)}$ for $x \in \mathbb{R}_+$, $k \in \mathbb{Z}$ and $c \in \mathbb{R}$ for which $\{\varphi_{c,k}\}_{k\in\mathbb{Z}} \in Y_c$ and $M^c(\varphi_{c,j};k) = 2\pi\delta_{j,k}$, $j,k \in \mathbb{Z}$, where $\delta_{j,k}$ is Kronecker's symbol. Moreover, $\{\varphi_{0,k}\}_{k\in\mathbb{Z}}$ for $c = 0$ is orthogonal with

$$\int_{e^{-\pi}}^{e^{\pi}} \varphi_{0,k}(u) \overline{\varphi_{0,j}(u)} \frac{du}{u} = 2\pi\delta_{j,k}.$$

Let $S_n^c(f;x)$ denote the partial sums of Mellin-Fourier series of f. Then

$$S_n^c(f;x) = \frac{x^{-c}}{2\pi} \sum_{k=-n}^{n} \mathcal{M}^c(f;k) x^{-ik}. \tag{1.2}$$

Using Mellin-Dirichlet kernel $D_n^c(x)$ which is given by

$$D_n^c(x) = \frac{x^{-c}}{2\pi}\sum_{k=-n}^{n} x^{-ik}, \quad x \in \mathbb{R}_+,$$

we can write $S_n^c(f;x)$ as finite Mellin convolution of f and D_n^c, i.e.,

$$S_n^c(f;x) = \int_{e^{-\pi}}^{e^{\pi}} D_n^c(u) f\left(\frac{x}{u}\right)\frac{du}{u}. \tag{1.3}$$

Using $x = e^{lnx}$, the relation $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and properties of trigonometric cosine series, we can obtain $D_n^c(x)$ as

$$D_n^c(x) = \begin{cases} \frac{2n+1}{2\pi}, & x = 1; \\ \frac{x^{-c}}{2\pi}\left(\frac{sin(n+1/2)lnx}{sin(lnx/2)}\right), & x \neq 1. \end{cases}$$

Therefore $S_n^c(f;x)$ can also be written as

$$\begin{aligned} S_n^c(f;x) &= \int_{e^{-\pi}}^{e^{\pi}} \frac{u^{-c}}{2\pi}\frac{sin((n+1)lnu)}{sin(lnu/2)} f\left(\frac{x}{u}\right)\frac{du}{u} \\ &= \frac{1}{2\pi}\int_{e^{-\pi}}^{e^{\pi}} \frac{sin((n+1)lnu)}{sin(lnu/2)} \tau_{1/u}^c(f;x)\frac{du}{u}, \end{aligned} \tag{1.4}$$

where $\tau_{1/u}$ being the Mellin translation operator defined by

$$\tau_h^c(f;x) = h^c f(hx), h \in \mathbb{R}_+.$$

The arithmetic means of the Mellin-Fourier series of $f \in Y_c$, denoted by $\sigma_n^c(f;x)$, are given by

$$\sigma_n^c(f;x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k^c(f;x), \quad n = 0, 1, 2, ... \tag{1.5}$$

which are knows as the $Cesáro$ means of order one, which are also referred as $(C, 1)$ means.

Let $a = \{a_n\}$ and $b = \{b_n\}$ be sequences of nonnegative integers with conditions

$$a_n < b_n, \quad n = 1, 2, 3, ... \quad \text{and} \quad \lim_{n\to\infty} b_n = \infty. \tag{1.6}$$

The Deferred $Cesáro$ mean is defined as follow

$$(D_{a,b}, s)_n := \frac{s_{a_n+1} + s_{a_n+2} + ... + s_{b_n}}{b_n - a_n} = \frac{1}{b_n - a_n}\sum_{k=a_n+1}^{b_n} s_k$$

where $\{s_k\}$ is a sequence of real or complex numbers. This method is regular under condition (1.6) ([2]).

2 Known Results

Lemma 2.1. *([6]) Let a function $g : [-\pi, \pi] \to \mathbb{R}$ be defined by*

$$g(t) = \begin{cases} \frac{sin(n+1/2)t}{sin(t/2)}, & t \neq 0\,; \\ 2n+1, & t = 0. \end{cases}$$

Then

$$\int_{-\pi}^{\pi} g(t)dt = 2\pi.$$

Lemma 2.2. *([5]) For $0 < \delta \leq \pi$*

$$\lim_{n\to\infty}\int_{\delta}^{\pi} \frac{sin(n+1/2)t}{sin(t/2)}dt = 0.$$

Theorem 2.3. *([4]) If $f \in Y_c$ for $c \in \mathbb{R}$, then*

$$\lim_{n\to\infty}\|\sigma_n^c(f;x) - f\|_{Y_c} = 0, \quad x \in \mathbb{R}_+.$$

3 Main Results

Let $\{p_n\}$ be a sequence of non-negative real numbers and

$$P_0^{b_n-a_n-1} = \sum_{k=0}^{b_n-a_n-1} p_k \neq 0, \quad P_{a_n+1}^{b_n} = \sum_{k=a_n+1}^{b_n} p_k \neq 0.$$

Then, Deferred Nörlund and Deferred Riesz means of $\{x_k\}$ are defined as follows

$$D_a^b N_n(x) := \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} x_k$$

and

$$D_a^b R_n(x) := \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k x_k,$$

respectively. Let $f \in Y_c$. Then, Deferred Nörlund and Deferred Riesz means of the series (1.1) are defined by

$$D_a^b N_n(f;x) := \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} S_k^c(f;x)$$

and

$$D_a^b R_n(f;x) := \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k S_k^c(f;x),$$

respectively.

Lemma 3.1. *Let $a = \{a_n\}$ and $b = \{b_n\}$ be sequences of nonnegative integers with conditions $a_n < b_n$, $b_n \to \infty$, $n \to \infty$ and $\{p_n\}$ be a non-increasing (non-decreasing) sequence of non-negative real numbers. Then for $t \in (0, \pi]$*

$$\sum_{k=0}^{b_n} p_k \frac{sin((k+1/2)t)}{sin(t/2)} \geq 0 \quad \left(\sum_{k=0}^{b_n} p_{b_n-k} \frac{sin((k+1/2)t)}{sin(t/2)} \geq 0 \right)$$

and

$$\sum_{k=0}^{a_n} p_k \frac{sin((k+1/2)t)}{sin(t/2)} \geq 0 \quad \left(\sum_{k=0}^{a_n} p_{b_n-k} \frac{sin((k+1/2)t)}{sin(t/2)} \geq 0 \right).$$

Proof. $t \in (0, \pi]$, $\left(\frac{sin(n+1/2)t}{sin(t/2)} \right)$ is a sequence of real numbers. Let

$$s_{b_n} = \sum_{k=0}^{b_n} \frac{sin((k+1/2)t)}{sin(t/2)}.$$

Then we have

$$\begin{aligned} s_{b_n} &= \frac{sin(t/2) + sin(3t/2) + ... + sin(b_n+1/2)t}{sint/2} . \frac{sint/2}{sint/2} \\ &= \frac{(cos0 - cost) + (cost - cos2t) + ... + (cos(b_n)t - cos(b_n+1)t)}{2sin^2(t/2)} \\ &= \frac{1 - cos(b_n+1)t}{2sin^2(t/2)} \\ &= \frac{2sin^2((b_n+1)t/2)}{2sin^2(t/2)} \\ &= \left(\frac{sin(b_n+1)t/2}{sint/2} \right)^2 \geq 0. \end{aligned}$$

By Abel's sum and hypothesis we get

$$\sum_{k=0}^{b_n} p_k \frac{sin((k+1/2)t)}{sin(t/2)} = s_{b_n}.p_{b_n} + \sum_{k=0}^{b_n-1} s_k(p_k - p_{k+1}) \geq 0.$$

In similar way we can show that $\sum_{k=0}^{a_n} p_k \frac{sin((k+1/2)t)}{sin(t/2)} \geq 0$. This is completed the proof. □

Theorem 3.2. *Let* $\{p_k\}$ *be a non-decreasing sequence of non-negative real numbers,* $\left(\frac{P_0^{b_n}}{P_0^{b_n-a_n-1}}\right)$ *and* $\left(\frac{P_{b_n-a_n}^{b_n}}{P_0^{b_n-a_n-1}}\right)$ *are bounded. If* $f \in Y_c$ *for* $c \in \mathbb{R}$*, then*

$$\lim_{n\to\infty} \|D_a^b N_n(f;x) - f\|_{Y_c} = 0, \quad x \in \mathbb{R}_+.$$

Proof. From Lemma 2.1 we have,

$$\begin{aligned}
D_a^b N_n(f;x) - f(x) &= \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} S_k^c(f;x) - f(x) \\
&= \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \int_{e^{-\pi}}^{e^{\pi}} \frac{u^{-c}}{2\pi} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} f\left(\frac{x}{u}\right) \frac{du}{u} \\
&- \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} f(x) \\
&= \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \int_{e^{-\pi}}^{e^{\pi}} \frac{u^{-c}}{2\pi} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} f\left(\frac{x}{u}\right) \frac{du}{u} \\
&- \frac{1}{2\pi} \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \int_{e^{-\pi}}^{e^{\pi}} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \frac{du}{u} f(x) \\
&= \frac{1}{2\pi} \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \int_{e^{-\pi}}^{e^{\pi}} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \left(u^{-c} f\left(\frac{x}{u}\right) - f(x)\right) \frac{du}{u} \\
&= \frac{1}{2\pi} \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \int_{e^{-\pi}}^{e^{\pi}} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} (\tau_{1/u}^c f(x) - f(x)) \frac{du}{u}.
\end{aligned}$$

Now, we get

$$\begin{aligned}
&\|D_a^b N_n(f;x) - f(x)\|_{Y_c} = \int_{e^{-\pi}}^{e^{\pi}} |D_a^b N_n(f;x)| x^{c-1} dx \\
&= \int_{e^{-\pi}}^{e^{\pi}} \left| \frac{1}{2\pi} \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \int_{e^{-\pi}}^{e^{\pi}} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} (\tau_{1/u}^c f(x) - f(x)) \frac{du}{u} \right| x^{c-1} dx \\
&\leq \frac{1}{2\pi} \int_{e^{-\pi}}^{e^{\pi}} \int_{e^{-\pi}}^{e^{\pi}} \left| \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \right| \left|\tau_{1/u}^c f(x) - f(x)\right| x^{c-1} dx \frac{du}{u} \\
&= \frac{1}{2\pi} \int_{e^{-\pi}}^{e^{\pi}} \left| \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \right| \int_{e^{-\pi}}^{e^{\pi}} \left|\tau_{1/u}^c f(x) - f(x)\right| x^{c-1} dx \frac{du}{u} \\
&= \frac{1}{2\pi} \int_{e^{-\pi}}^{e^{\pi}} \left| \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \right| \|\tau_{1/u}^c f - f\|_{Y_c} \frac{du}{u}.
\end{aligned}$$

Let $E_\delta := \{x \in [e^{-\pi}, e^{\pi}] : |x-1| < \delta\}$, $0 \le \delta < 1-e^{-\pi}$ and $CE_\delta = [e^{-\pi}, e^{\pi}] - E_\delta$. Then,

$$\begin{aligned}&\|D_a^b N_n(f;x) - f(x)\|_{Y_c}\\ =\ &\frac{1}{2\pi}\left\{\int_{E_\delta} + \int_{CE_\delta}\right\}\left|\frac{1}{P_0^{b_n-a_n-1}}\sum_{a_n+1}^{b_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\|\tau_{1/u}^c f - f\|_{Y_c}\frac{du}{u}.\end{aligned}$$

Since $\underset{h\to 1}{lim}\|\tau_h^c f - f\|_{Y_c} = 0$, so for a given $\epsilon > 0$, $\exists\delta(0 \le \delta < 1-e^{-\pi})$ such that $\forall u \in E_\delta$, $\|\tau_{1/u}^c f - f\|_{Y_c} < \frac{\epsilon}{2}$. Also $\|\tau_{1/u}^c f - f\|_{Y_c} \le 2\|f\|_{Y_c}$. So,

$$\begin{aligned}\|D_a^b N_n(f;x) - f(x)\|_{Y_c} &= \frac{1}{2\pi}\int_{E_\delta}\left|\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=a_n+1}^{b_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{\epsilon}{2}\frac{du}{u}\\ &+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\left|\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=a_n+1}^{b_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u}\\ &\le \frac{1}{2\pi}\int_{e^{-\pi}}^{e^{\pi}}\left|\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=a_n+1}^{b_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{\epsilon}{2}\frac{du}{u}\\ &+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\left|\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=a_n+1}^{b_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u}\\ &\le \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\left|\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=0}^{b_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u}\\ &+ \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\left|\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=0}^{a_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u}\\ &+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\left|\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=0}^{b_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u}\\ &+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\left|\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=0}^{a_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u}.\end{aligned}$$

By Lemma 3.1 and hypothesis, we obtain

$$\begin{aligned}\|D_a^b N_n(f;x) - f(x)\|_{Y_c} &\le \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\frac{P_0^{b_n}}{P_0^{b_n-a_n-1}}\frac{1}{P_0^{b_n}}\sum_{k=0}^{b_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}\\ &+ \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\frac{P_{b_n-a_n}^{b_n}}{P_0^{b_n-a_n-1}}\frac{1}{P_{b_n-a_n}^{b_n}}\sum_{k=0}^{a_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}\\ &+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\frac{P_0^{b_n}}{P_0^{b_n-a_n-1}}\frac{1}{P_0^{b_n}}\sum_{k=0}^{b_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}\\ &+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\frac{P_{b_n-a_n}^{b_n}}{P_0^{b_n-a_n-1}}\frac{1}{P_{b_n-a_n}^{b_n}}\sum_{k=0}^{a_n} p_{b_n-k}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}} \frac{1}{P_0^{b_n}} \sum_{k=0}^{b_n} p_{b_n-k} \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}} \frac{1}{P_{b_n-a_n}^{b_n}} \sum_{k=0}^{a_n} p_{b_n-k} \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta} \frac{1}{P_0^{b_n}} \sum_{k=0}^{b_n} p_{b_n-k} \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta} \frac{1}{P_{b_n-a_n}^{b_n}} \sum_{k=0}^{a_n} p_{b_n-k} \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}
\end{aligned}
$$

So, we get from Lemma 2.2,

$$
\begin{aligned}
\lim_{n\to\infty}\|D_a^b N_n(f;x)-f(x)\|_{Y_c} &\leq \epsilon + \lim_{n\to\infty}\frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta} \frac{1}{P_0^{b_n}} \sum_{k=0}^{b_n} p_{b_n-k} \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \lim_{n\to\infty}\frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta} \frac{1}{P_0^{a_n}} \sum_{k=0}^{a_n} p_{b_n-k} \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&= \epsilon + \frac{\|f\|_{Y_c}}{\pi}\frac{1}{P_0^{b_n}} \sum_{k=0}^{b_n} p_{b_n-k} \lim_{n\to\infty}\int_{CE_\delta} \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\frac{1}{P_0^{a_n}} \sum_{k=0}^{a_n} p_{b_n-k} \lim_{n\to\infty}\int_{CE_\delta} \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&= \epsilon + \frac{\|f\|_{Y_c}}{\pi}(0+0) \\
&= \epsilon.
\end{aligned}
$$

Thus

$$
\lim_{n\to\infty}\|D_a^b N_n(f;x)-f(x)\|_{Y_c} = 0.
$$

This completes the proof.

Theorem 3.3. *Let* $\{p_k\}$ *be a non-increasing sequence of non-negative real numbers,* $\left(\frac{P_0^{a_n}}{P_{a_n+1}^{b_n}}\right)$ *and* $\left(\frac{P_0^{b_n}}{P_{a_n+1}^{b_n}}\right)$ *are bounded. If* $f \in Y_c$ *for* $c \in \mathbb{R}$, *then*

$$
\lim_{n\to\infty}\|D_a^b R_n(f;x)-f\|_{Y_c} = 0, \quad x \in \mathbb{R}_+.
$$

Proof. From Lemma 2.1 we have,

$$
\begin{aligned}
D_a^b R_n(f;x) - f(x) &= \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k S_k^c(f;x) - f(x) \\
&= \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \int_{e^{-\pi}}^{e^{\pi}} \frac{u^{-c}}{2\pi} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} f\left(\frac{x}{u}\right) \frac{du}{u} \\
&- \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k f(x) \\
&= \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \int_{e^{-\pi}}^{e^{\pi}} \frac{u^{-c}}{2\pi} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} f\left(\frac{x}{u}\right) \frac{du}{u} \\
&- \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi} \int_{e^{-\pi}}^{e^{\pi}} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \frac{du}{u} f(x) \\
&= \frac{1}{2\pi} \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \int_{e^{-\pi}}^{e^{\pi}} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \left(u^{-c} f\left(\frac{x}{u}\right) - f(x)\right) \frac{du}{u} \\
&= \frac{1}{2\pi} \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \int_{e^{-\pi}}^{e^{\pi}} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} (\tau_{1/u}^c f(x) - f(x)) \frac{du}{u}.
\end{aligned}
$$

Now, we get

$$
\begin{aligned}
&\|D_a^b R_n(f;x) - f(x)\|_{Y_c} = \int_{e^{-\pi}}^{e^{\pi}} |D_a^b R_n(f;x)| x^{c-1} dx \\
&= \int_{e^{-\pi}}^{e^{\pi}} \left| \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \int_{e^{-\pi}}^{e^{\pi}} \frac{sin((k+1/2)lnu)}{sin(lnu/2)} (\tau_{1/u}^c f(x) - f(x)) \frac{du}{u} \right| x^{c-1} dx \\
&\leq \frac{1}{2\pi} \int_{e^{-\pi}}^{e^{\pi}} \int_{e^{-\pi}}^{e^{\pi}} \left| \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \right| \left| \tau_{1/u}^c f(x) - f(x) \right| x^{c-1} dx \frac{du}{u} \\
&= \frac{1}{2\pi} \int_{e^{-\pi}}^{e^{\pi}} \left| \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \right| \int_{e^{-\pi}}^{e^{\pi}} \left| \tau_{1/u}^c f(x) - f(x) \right| x^{c-1} dx \frac{du}{u} \\
&= \frac{1}{2\pi} \int_{e^{-\pi}}^{e^{\pi}} \left| \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \right| \|\tau_{1/u}^c f - f\|_{Y_c} \frac{du}{u}.
\end{aligned}
$$

Let $E_\delta := \{x \in [e^{-\pi}, e^{\pi}] : |x-1| < \delta\}$, $0 \leq \delta < 1 - e^{-\pi}$ and $CE_\delta = [e^{-\pi}, e^{\pi}] - E_\delta$. Then

$$
\|D_a^b R_n(f;x) - f(x)\|_{Y_c} = \frac{1}{2\pi} \left\{ \int_{E_\delta} + \int_{CE_\delta} \right\} \left| \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)} \right| \|\tau_{1/u}^c f - f\|_{Y_c} \frac{du}{u}.
$$

Since $\lim_{h\to 1} \|\tau_h^c f - f\|_{Y_c} = 0$, so for a given $\epsilon > 0$, $\exists \delta (0 \leq \delta < 1 - e^{-\pi})$ such that $\forall u \in E_\delta$,

$\|\tau^c_{1/u}f - f\|_{Y_c} < \frac{\epsilon}{2}$. Also $\|\tau^c_{1/u}f - f\|_{Y_c} \le 2\|f\|_{Y_c}$. So

$$\begin{aligned}
\|D^b_a R_n(f;x) - f(x)\|_{Y_c} &= \frac{1}{2\pi}\int_{E_\delta}\left|\frac{1}{P^{b_n}_{a_n+1}}\sum_{k=a_n+1}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{\epsilon}{2}\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\left|\frac{1}{P^{b_n}_{a_n+1}}\sum_{k=a_n+1}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u} \\
&\le \frac{1}{2\pi}\int_{e^{-\pi}}^{e^{\pi}}\left|\frac{1}{P^{b_n}_{a_n+1}}\sum_{k=a_n+1}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{\epsilon}{2}\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\left|\frac{1}{P^{b_n}_{a_n+1}}\sum_{k=a_n+1}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u} \\
&\le \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\left|\frac{1}{P^{b_n}_{a_n+1}}\sum_{k=0}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u} \\
&+ \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\left|\frac{1}{P^{b_n}_{a_n+1}}\sum_{k=0}^{a_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\left|\frac{1}{P^{b_n}_{a_n+1}}\sum_{k=0}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\left|\frac{1}{P^{b_n}_{a_n+1}}\sum_{k=0}^{a_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\right|\frac{du}{u}.
\end{aligned}$$

From Lemma 3.1 and hypothesis, we obtain

$$\begin{aligned}
\|D^b_a R_n(f;x) - f(x)\|_{Y_c} &\le \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\frac{P^{b_n}_0}{P^{b_n}_{a_n+1}}\frac{1}{P^{b_n}_0}\sum_{k=0}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\frac{P^{a_n}_0}{P^{b_n}_{a_n+1}}\frac{1}{P^{a_n}_0}\sum_{k=0}^{a_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\frac{P^{b_n}_0}{P^{b_n}_{a_n+1}}\frac{1}{P^{b_n}_0}\sum_{k=0}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\frac{P^{a_n}_0}{P^{b_n}_{a_n+1}}\frac{1}{P^{a_n}_0}\sum_{k=0}^{a_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&\le \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\frac{1}{P^{b_n}_0}\sum_{k=0}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\epsilon}{4\pi}\int_{e^{-\pi}}^{e^{\pi}}\frac{1}{P^{a_n}_0}\sum_{k=0}^{a_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\frac{1}{P^{b_n}_0}\sum_{k=0}^{b_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u} \\
&+ \frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\frac{1}{P^{a_n}_0}\sum_{k=0}^{a_n} p_k \frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}
\end{aligned}$$

So, we get from Lemma 2.2,

$$\begin{aligned}
\lim_{n\to\infty}\|D_a^bR_n(f;x)-f(x)\|_{Y_c} &\leq \epsilon+\lim_{n\to\infty}\frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\frac{1}{P_0^{b_n}}\sum_{k=0}^{b_n}p_k\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}\\
&+\lim_{n\to\infty}\frac{\|f\|_{Y_c}}{\pi}\int_{CE_\delta}\frac{1}{P_0^{a_n}}\sum_{k=0}^{a_n}p_k\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}\\
&= \epsilon+\frac{\|f\|_{Y_c}}{\pi}\frac{1}{P_0^{b_n}}\sum_{k=0}^{b_n}p_k\lim_{n\to\infty}\int_{CE_\delta}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}\\
&+\frac{\|f\|_{Y_c}}{\pi}\frac{1}{P_0^{a_n}}\sum_{k=0}^{a_n}p_k\lim_{n\to\infty}\int_{CE_\delta}\frac{sin((k+1/2)lnu)}{sin(lnu/2)}\frac{du}{u}\\
&= \epsilon+\frac{\|f\|_{Y_c}}{\pi}(0+0)\\
&= \epsilon.
\end{aligned}$$

Thus

$$\lim_{n\to\infty}\|D_a^bR_n(f;x)-f(x)\|_{Y_c}=0.$$

This completes the proof. □

In case $p_k = 1$ for all k, the methods $D_a^bN_n(f;x)$ and $D_a^bR_n(f;x)$ give us classically well-known Deferred $Ces\acute{a}ro$ means.

Corollary 3.4. *Let* $\left(\frac{a_n+1}{b_n-a_n}\right)$ *and* $\left(\frac{b_n+1}{b_n-a_n}\right)$ *are bounded. Then, if* $f\in Y_c$ *for* $c\in\mathbb{R}$ *we have*

$$\lim_{n\to\infty}\|D_a^b(f;x)-f\|_{Y_c}=0,\quad x\in\mathbb{R}_+.$$

The proof is similar to that of Theorem 3.2 and Theorem 3.3. So, we omit details.

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