

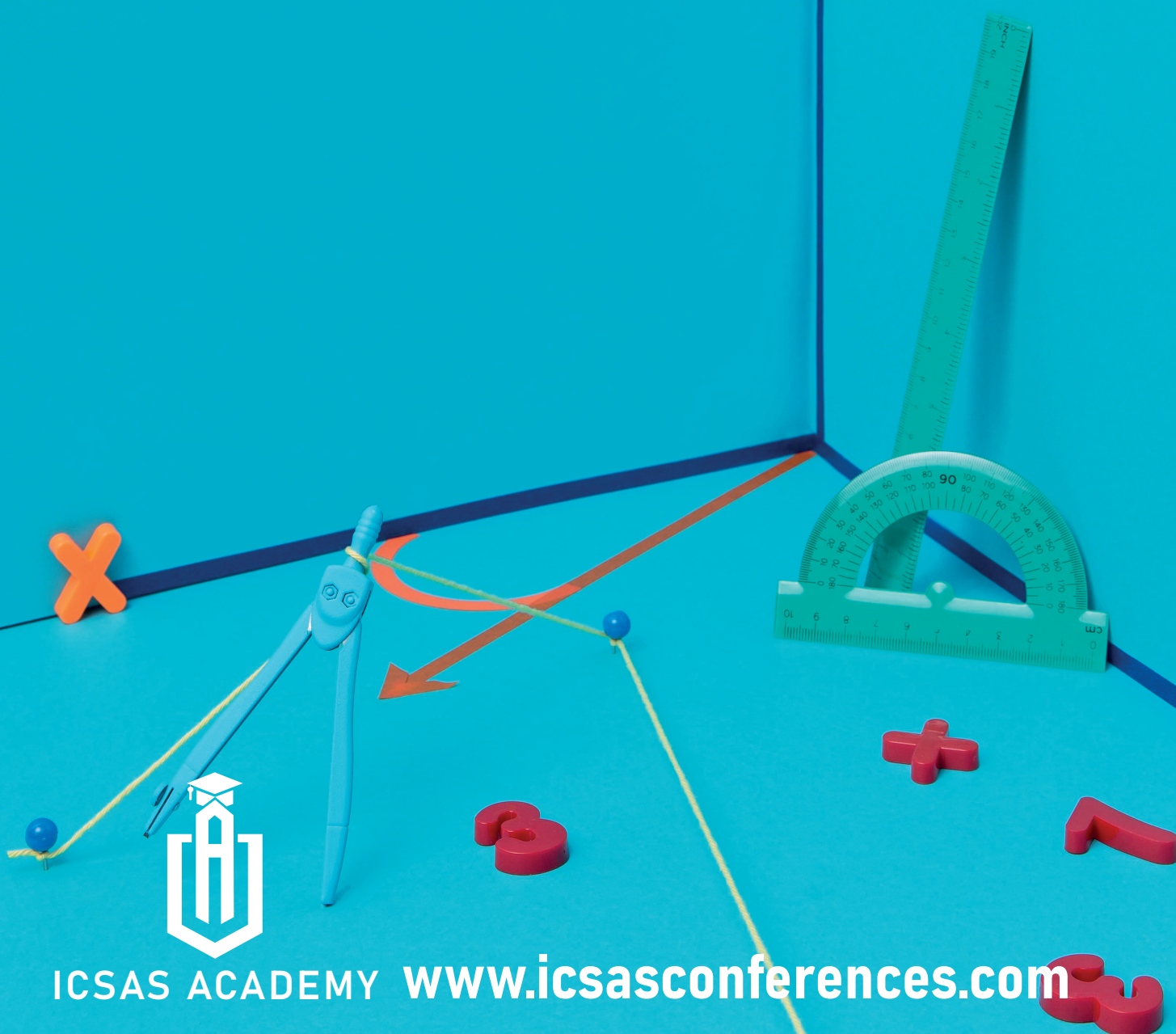
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September 19 - 21, 2025

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		2	GEOMETRIC PROPERTIES OF PARA-KENMOTSU MANIFOLDS ADMITTING -RICCI-BOURGUIGNON SOLITONS	DOÇ. DR. TUĞBA MERT PROF. DR. MEHMET ATÇEKEN
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CHROMATIC ZAGREB INDEX OF ZERO DIVISOR GRAPHS**Assistant Professor Dr., ÖZGE ÇOLAKOĞLU**Mersin University, ozgeclkg@gmail.com - ORCID ID: [0000-0003-4094-3380](https://orcid.org/0000-0003-4094-3380)**ÖZET**

Algebraic graph theory has become an important field due to its applications in chemistry, cryptology, and information sciences. In this study, we focus on the chromatic first Zagreb index, a recently introduced topological graph invariant that extends the classical Zagreb index by incorporating vertex colorings. While the classical Zagreb index depends solely on the degrees of vertices, its chromatic counterpart provides a more refined structural measure by considering graph colorings and their associated parameters. The main objective of this paper is to investigate the chromatic first Zagreb indices for zero-divisor graphs of commutative rings of the form $\mathbb{Z}_{\xi^k}$ ($k \geq 2$), $\mathbb{Z}_{\xi\rho}$, $\mathbb{Z}_{\xi^2\rho}$ and $\mathbb{Z}_{\xi^2\rho^2}$ where ξ and ρ denote prime numbers. Explicit formulas for both the minimum and maximum chromatic first Zagreb indices are derived for these families of graphs. Furthermore, we compare the obtained results with the classical first Zagreb indices, highlighting significant differences and growth rates depending on the algebraic structure under consideration.

The analysis reveals that for certain cases, such as $\Gamma(\mathbb{Z}_4)$, $\Gamma(\mathbb{Z}_6)$, $\Gamma(\mathbb{Z}_9)$ and $\Gamma(\mathbb{Z}_{16})$ the chromatic first Zagreb index can exceed the classical Zagreb index, while in most other cases the classical Zagreb index dominates. Additionally, it is shown that the values of $CM_{1,t}^-$ are always smaller than those of $CM_{1,t}^+$ and M_1 confirming the non-triviality of chromatic variations. For $k \geq 3$, the growth rate of the chromatic Zagreb index surpasses that of the classical index, whereas for rings such as $\mathbb{Z}_{\xi^2\rho^2}$ the opposite behavior is observed.

These findings provide new insights into the interplay between algebraic ring structures and graph invariants. The results not only enrich the theoretical framework of graph indices but also suggest potential applications in cryptology, algorithm design, and chemical graph theory, where algebraic structures are frequently employed to model complex systems.

Anahtar Kelimeler : Graph Thoery, Topological index, Chromatic first Zagreb index, Zero-divisor graphs of commutative rings.

1. INTRODUCTION

In the rapidly developing pharmaceutical industry, there is an increasing need to predict chemical structure properties without experimental procedures. Graph indices, a fundamental component of chemical graph theory since 1947 [16], address this need by providing mathematical representations of chemical structures. Among these, the Zagreb index which is degree-dependent has garnered significant attention.

The first Zagreb index of a graph G with vertex set V and edge set E is defined as follows [9]:

$$M_1(G) = \sum_{w \in V} d_w^2$$

where d_w represents the degree of vertex w .

Recently, graph indices based on vertex coloring have been introduced, extending their applicability to problems involving graph models of daily life scenarios [10]. Proper vertex coloring assigns distinct colors to adjacent vertices, effectively labeling vertices with integers. Let N be positive integers and coloring be a function. In that case, $\varsigma: V \rightarrow \mathbb{N}$ is a function such that $\varsigma(u) \neq \varsigma(v)$ for $uv \in E$. If a graph G is colored by the γ -minimum positive integer, then the graph G is γ -coloring [6]. A graph G is labeled with γ -coloring by the cardinality $\gamma!$. Kok et al. [10] defined chromatic Zagreb indices as follows:

Definition 1.1. Let G be a graph γ -coloring such that $\varsigma(\varpi) \in N$ and $1 \leq \varsigma(\varpi) \leq \gamma$ for $\varpi \in V$. Therefore when $1 \leq t \leq \gamma!$, the chromatic first Zagreb index of G [10]:

$$CM_{1,t}(G) = \sum_{w \in V} \varsigma(w)^2$$

The minimum and maximum chromatic first Zagreb indices are defined as

$$CM_{1,t}^-(G) = \min \{CM_{1,t}(G): 1 \leq t \leq \gamma!\}$$

$$CM_{1,t}^+(G) = \max \{CM_{1,t}(G): 1 \leq t \leq \gamma!\}$$

The first work on combining algebraic structures with graph theory is Cayley graphs of finite groups, introduced by Arthur Cayley [5]. Algebraic graph structures are used in many areas. Dong et al. suggested robot design with the subgroups of dihedral group [7]. Kotorowicz and Ustimenko worked on cryptoalgorithms with algebraic graphs [11]. Shaska and Ustimenko studied on applications of coding theory and cryptography [14].

Zero-divisor graphs, first introduced by Beck [4] and refined by Anderson and Livingston [3], are another prominent class. These graphs $\Gamma(R)$ represent elements of commutative rings R , where vertices correspond to non-zero zero divisors and edges connect vertices whose product is a zero divisor. Throughout this study, q, p, p_1, p_2, q are prime numbers.

Rayer and Jeyaraj studied eccentric based indices of a commutative ring [13]. The eccentric index based indices of $Z_{p_1 p_2} \times Z_q$ and $Z_{p^2} \times Z_q$ were studied [8]. Ahmadi and Nezaad studied some indices of Z_{pq} and Z_{p^2} [1]. Alali et al. given results on M- polynomials of some algebraic graphs [2]. Mazlan et al. found first Zagreb index of $\Gamma(Z_p)$ [12]. Semil et al. studied first Zagreb index of $\Gamma(Z_{p^k})$ [15].

This paper focuses on the chromatic first Zagreb indices for the zero-divisor graphs of $\mathbb{Z}_{\xi^k}$ ($k \geq 2$), $\mathbb{Z}_{\xi\rho}$, $\mathbb{Z}_{\xi^2\rho}$ and $\mathbb{Z}_{\xi^2\rho^2}$. The results are compared with classical Zagreb indices, emphasizing their potential utility in cryptology and algorithm analysis.

2. Chromatic Index of Zero-Divisor Graphs

In this section, chromatic first Zagreb indices $\mathbb{Z}_{\xi^k}$ ($k \geq 2$), $\mathbb{Z}_{\xi\rho}$, $\mathbb{Z}_{\xi^2\rho}$ and $\mathbb{Z}_{\xi^2\rho^2}$ zero-divisor graph are found.

Theorem 2.1. If $q = 2$ then $CM_{1,t}^-(\Gamma(\mathbb{Z}_{q^2})) = CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^2})) = 3$

Proof. When $q = 2$, the ring $R = \mathbb{Z}_{q^2}$ has elements $\{1, 2, 3\}$, and the graph $\Gamma(\mathbb{Z}_{q^2})$ has an empty edge set. The chromatic number is 1, resulting in all vertices labeled with the same color. Since $\gamma = 1$, Using Eq. (1.2), (1.3) and (1.4), it have

$$CM_{1,t}^-(\Gamma(\mathbb{Z}_{q^2})) = CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^2})) = 1^2 + 1^2 + 1^2 = 3.$$

Corollary 2.1. For $q = 2$, $M_1(\Gamma(\mathbb{Z}_{q^2})) = 0 < CM_{1,t}^-(\Gamma(\mathbb{Z}_{q^2})) = CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^2}))$.

Proof. When $q = 2$, the ring $R = \mathbb{Z}_{q^2}$ has elements $\{1, 2, 3\}$, and the graph $\Gamma(\mathbb{Z}_{q^2})$ has an empty edge set. The chromatic number is 1, resulting in all vertices labeled with the same color. Since $\gamma = 1$, Using Eq. (1.2), (1.3) and (1.4), it have

$$CM_{1,t}^-(\Gamma(\mathbb{Z}_{q^2})) = CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^2})) = 1^2 + 1^2 + 1^2 = 3.$$

Theorem 2.2. If $\varrho \geq 3$ then $CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2})) = CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2})) = \frac{\varrho(\varrho-1)(2\varrho-1)}{6}$.

Proof. Let $\varrho \geq 3$. In that case, $\Gamma(\mathbb{Z}_{\varrho^2}) \cong K_{\varrho-1}$ and $|V(\Gamma(\mathbb{Z}_{\varrho^2}))| = \varrho - 1$. Therefore, each vertex is assigned with a different color and so this graph is $(\varrho - 1)$ -coloring. Then, $\gamma = \varrho - 1$. From Eq. (1.2),

$$CM_{1,t}(\Gamma(\mathbb{Z}_{\varrho^2})) = \sum_{w \in V} \varsigma(w)^2 = \sum_{i=1}^{\varrho-1} i^2 = \frac{\varrho(\varrho-1)(2\varrho-1)}{6}.$$

From Eq. (1.3) and (1.4), the result is obtained.

Figure 2 shows plot of $CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2}))$, $CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2}))$ and $M_1(\Gamma(\mathbb{Z}_{\varrho^2}))$ for $\varrho \geq 3$.

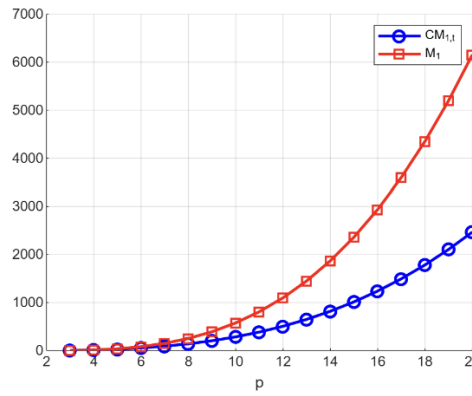


Figure 1. Plot of $CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2}))$, $CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2}))$ and $M_1(\Gamma(\mathbb{Z}_{\varrho^2}))$ for $\varrho \geq 3$

Corollary 2.2. If $\varrho = 3$ or $\varrho = 4$ then $d(w) = \varrho - 2$ for $w \in V$ and

$$M_1(\Gamma(\mathbb{Z}_{\varrho^2})) = (\varrho - 1)(\varrho - 2)^2 < CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2})) = CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2})).$$

For $\varrho \geq 5$,

$$M_1(\Gamma(\mathbb{Z}_{\varrho^2})) = (\varrho - 1)(\varrho - 2)^2 > CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2})) = CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2})).$$

Theorem 2.3. If $\varrho \geq 3$ and $\xi \geq 3$, then

$$CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^\xi})) = \varrho^{\xi+k}(\varrho^{k-1} - 1) + \frac{\varrho^k(\varrho^k - 1)(2\varrho^k - 1)}{6}$$

$$CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^\xi})) = \varrho^{\xi-k}(\varrho^{k-1} - 1) + \frac{\varrho^k(\varrho^k - 1)(2\varrho^k - 1)}{6} - 1$$

where ξ is odd then $k = \frac{\xi+1}{2}$, otherwise $k = \frac{\xi}{2}$.

Proof. $V(\Gamma(\mathbb{Z}_{\varrho^\xi}))$ can be separated as follows

$$U_1 = \{\varrho\lambda | 1 \leq \lambda \leq \varrho^{\xi-1} - 1, \varrho \nmid \lambda\}$$

$$U_2 = \{\varrho^2\lambda | 1 \leq \lambda \leq \varrho^{\xi-2} - 1, \varrho \nmid \lambda\}$$

for $i = 3, 4, \dots, \xi - 2$

$$\mathcal{U}_i = \{q^i \lambda \mid 1 \leq \lambda \leq q^{\xi-i} - 1, q \nmid \lambda\}$$

and

$$\mathcal{U}_{\xi-1} = \{q^{\xi-1} \lambda \mid 1 \leq \lambda \leq q - 1, q \nmid \lambda\}$$

It have that $|\mathcal{U}_i| = q^{\xi-i-1}(q - 1)$ for $i = 1, 2, \dots, \xi - 1$. If ξ is odd then $k = \frac{\xi+1}{2}$, otherwise $k = \frac{\xi}{2}$.

Let $\sim$ Show that there is a adjacent relationship.

$$\mathcal{U}_1 \sim \mathcal{U}_{\xi-1}$$

$$\mathcal{U}_2 \sim \mathcal{U}_{\xi-2}, \mathcal{U}_2 \sim \mathcal{U}_{\xi-1}$$

...

$$\mathcal{U}_{i-1} \sim \mathcal{U}_j \text{ for } i < k \text{ and } j = i + 1, \dots, \xi - 1$$

$$\mathcal{U}_i \sim \mathcal{U}_i, \mathcal{U}_i \sim \mathcal{U}_j \text{ for } i \geq k \text{ and } j = i + 1, \dots, \xi - 1.$$

Since $\mathcal{U}_1 \sim \mathcal{U}_1$, all vertices in $\mathcal{U}_1$ are assigned the same color (labeled with the same number). Likewise, if $i < k$ then $\mathcal{U}_i \sim \mathcal{U}_i$ and all vertices in $\mathcal{U}_i$ are assigned the same color. If $\mathcal{U}_i \sim \mathcal{U}_i$ then all vertices in $\mathcal{U}_i$ are assigned $|\mathcal{U}_i|$ different colors. Then, for $1 < i < k$, all vertices in $\mathcal{U}_i$ are assigned the same colors. For $\xi-1 \geq i \geq k$, all vertices in $\mathcal{U}_i$ are assigned the different colors. Then, for coloring vertices of this graph are needed $\sum_{i=k}^{\xi-1} |\mathcal{U}_i| + 1$. That is,

$$\gamma = \sum_{i=k}^{\xi-1} |\mathcal{U}_i| + 1 = \sum_{i=k}^{\xi-1} (q^{\xi-i-1}(q - 1)) + 1 = (q - 1) \sum_{i=k}^{\xi-1} q^i + 1 = q^k$$

From Eq. (1.2), it can be the following equation:

$$\begin{aligned} CM_{1,t}^-(\Gamma(\mathbb{Z}_{q^\xi})) &= |\mathcal{U}_1|1^2 + \dots + |\mathcal{U}_{k-1}|1^2 + \sum_{i=2}^{\gamma} i^2 \\ &= \sum_{i=1}^k |\mathcal{U}_i| + \sum_{i=2}^{\gamma} i^2 = (q - 1) \sum_{i=2}^k q^{\xi-i} + \frac{\gamma(\gamma + 1)(2\gamma + 1)}{6} - 1 \\ &= q^\xi(q - 1) \frac{1 - q^{k-1}}{q^k(1 - q)} + \frac{q^k(q^k + 1)(2q^k + 1)}{6} - 1. \end{aligned}$$

If $\varpi \in \mathcal{U}_j$, $1 < j < k$ then $\varsigma(\varpi) = \gamma$ from Eq. (1.4). All other vertices are assigned different numbers. Using Eq. (1.2),

$$\begin{aligned} CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^\xi})) &= |\mathcal{U}_1|\gamma^2 + \dots + |\mathcal{U}_{k-1}|\gamma^2 + \sum_{i=1}^{\gamma-1} i^2 \\ &= \gamma^2 \sum_{i=1}^{k-1} |\mathcal{U}_i| + \sum_{i=1}^{\gamma-1} i^2 = q^{2k} q^\xi (q - 1) \frac{1 - q^{k-1}}{q^k(1 - q)} + \frac{\gamma(\gamma - 1)(2\gamma - 1)}{6} \end{aligned}$$

$$= \varrho^{2k} \varrho^\xi (\varrho - 1) \frac{1 - \varrho^{k-1}}{\varrho^k (1 - \varrho)} + \frac{\varrho^k (\varrho^k - 1)(2\varrho^k - 1)}{6}.$$

Corollary 2.3. Since $d(w) = \varrho - 1$ for $w \in \mathcal{U}_1$, $d(w) = \varrho^2 - 1$ for $w \in \mathcal{U}_2$, $d(w) = \varrho^i - 1$ for $w \in \mathcal{U}_i$ when $1 < j < k$, $d(w) = \varrho^i - 2$ for $w \in \mathcal{U}_i$ when $\xi - 1 \geq i \geq k$, the following inequality is obtained:

$$CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^\xi})) < M_1(\Gamma(\mathbb{Z}_{\varrho^\xi})) < CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^\xi}))$$

$$\text{where } M_1(\Gamma(\mathbb{Z}_{\varrho^\xi})) = \varrho^\xi (\varrho - 1) \left(\sum_{i=1}^{k-1} \frac{(\varrho^i - 1)^2}{\varrho^{i+1}} + \sum_{i=k}^{\xi-1} \frac{(\varrho^i - 2)^2}{\varrho^{i+1}} \right).$$

Figure 2 shows plot of $CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^\xi}))$, $M_1(\Gamma(\mathbb{Z}_{\varrho^\xi}))$, $CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^\xi}))$ for $\xi = 5$.

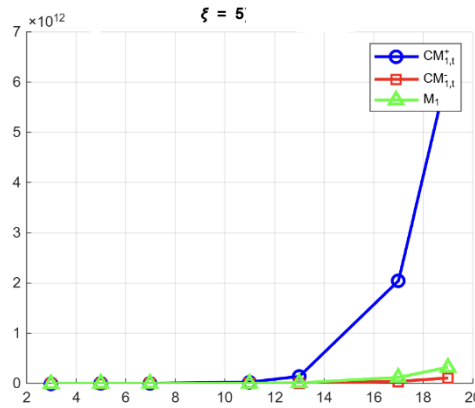


Figure 2. The plot of $CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^\xi}))$, $M_1(\Gamma(\mathbb{Z}_{\varrho^\xi}))$, $CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^\xi}))$

Theorem 2.4. Let $\Gamma(\mathbb{Z}_{\varrho q})$ be graph of $\mathbb{Z}_{\varrho q}$ non-zero zero divisor rings. If $q > \varrho$ then,

$$CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho q})) = 4\varrho + q - 5,$$

$$CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho q})) = 4q + \varrho - 5.$$

Proof. $V(\Gamma(\mathbb{Z}_{\varrho q}))$ can be divided:

$$\mathcal{U}_1 = \{\varrho\lambda | 1 \leq \lambda \leq q - 1, \varrho \nmid \lambda\}$$

$$\mathcal{U}_2 = \{q\lambda | 1 \leq \lambda \leq \varrho - 1, q \nmid \lambda\}$$

$\Gamma(\mathbb{Z}_{\varrho q})$ graph is complete bipartite graph. $\mathcal{U}_1 \sim \mathcal{U}_2$, $\mathcal{U}_1 \not\sim \mathcal{U}_1$ and $\mathcal{U}_2 \not\sim \mathcal{U}_2$. So, the vertices of this graph are labeled with 1 and 2. Then, $\gamma = 2$. Since $q > \varrho$, $|\mathcal{U}_1| = q - 1 > |\mathcal{U}_2| = \varrho - 1$. From Eq. (1.2) and (1.3),

$$\begin{aligned} CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho q})) &= \sum_{w \in \mathcal{U}_1} \varsigma(w)^2 + \sum_{w \in \mathcal{U}_2} \varsigma(w)^2 = |\mathcal{U}_1|1^2 + |\mathcal{U}_2|2^2 \\ &= (q - 1) + (\varrho - 1)4 = 4\varrho + q - 5. \end{aligned}$$

From Eq. (1.2) and (1.4), the following equation is obtained:

$$\begin{aligned} CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho q})) &= \sum_{w \in \mathcal{U}_1} \varsigma(w)^2 + \sum_{w \in \mathcal{U}_2} \varsigma(w)^2 = |\mathcal{U}_1|2^2 + |\mathcal{U}_2|1^2 \\ &= 4(q-1) + (\varrho-1) = 4q + \varrho - 5. \end{aligned}$$

Corollary 2.4. $d(w) = \varrho - 1$ for $w \in \mathcal{U}_1$ and $d(w) = q - 1$ for $w \in \mathcal{U}_2$ because $\Gamma(\mathbb{Z}_{\varrho q})$ is complete bipartite graph. Let $\varrho = 2$ and $q = 3$. Then

$$M_1(\Gamma(\mathbb{Z}_{\varrho q})) = 6 = CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho q})) < CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho q})).$$

Otherwise,

$$M_1(\Gamma(\mathbb{Z}_{\varrho q})) = (\varrho - 1)(q - 1)(\varrho + q - 2) > CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho q})) > CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho q})).$$

Figure 3 shows plot of $M_1(\Gamma(\mathbb{Z}_{\varrho q}))$, $CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho q}))$, $CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho q}))$ for $q > \varrho$.

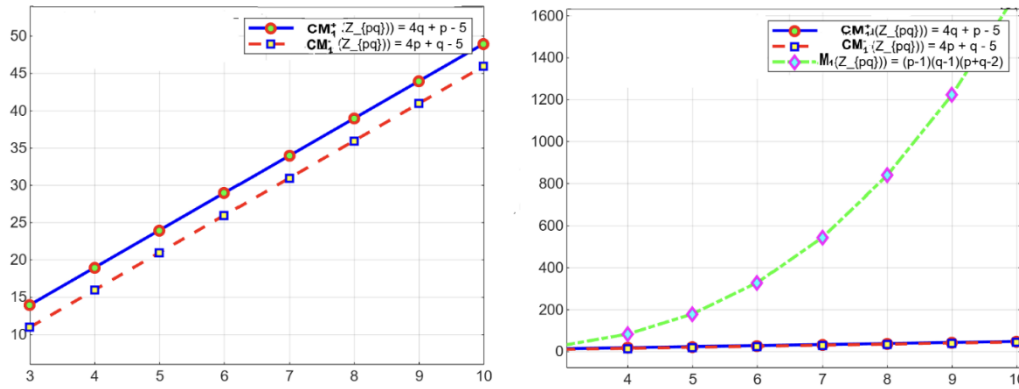


Figure 3. The plot of $M_1(\Gamma(\mathbb{Z}_{\varrho q}))$, $CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho q}))$, $CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho q}))$ for $q > \varrho$.

Theorem 2.5. Let $\Gamma(\mathbb{Z}_{\varrho^2 q})$ be graph of $\mathbb{Z}_{\varrho^2 q}$ non-zero zero divisor rings. If $q > \varrho$ then,

$$CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2 q})) = (q-1)\varrho^3 + \varrho(\varrho-1)^3 + \frac{\varrho(\varrho-1)(2\varrho-1)}{6},$$

$$CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2 q})) = \varrho q + \frac{9}{2}\varrho^2 + \frac{\varrho^3}{3} - \frac{29\varrho}{6} - 1.$$

Proof. $V(\Gamma(\mathbb{Z}_{\varrho^2 q}))$ can be divided:

$$\mathcal{U}_1 = \{\varrho\lambda | 1 \leq \lambda \leq \varrho q - 1, \varrho \nmid \lambda, q \nmid \lambda\}, |\mathcal{U}_1| = (q-1)(\varrho-1)$$

$$\mathcal{U}_2 = \{q\lambda | 1 \leq \lambda \leq \varrho^2 - 1, \varrho \nmid \lambda\}, |\mathcal{U}_2| = \varrho(\varrho-1)$$

$$\mathcal{U}_3 = \{\varrho^2\lambda | 1 \leq \lambda \leq q-1, \varrho \nmid \lambda, q \nmid \lambda\}, |\mathcal{U}_3| = (q-1)$$

$$\mathcal{U}_4 = \{\varrho q\lambda | 1 \leq \lambda \leq \varrho-1, q \nmid \lambda\}, |\mathcal{U}_4| = (\varrho-1)$$

As can be seen from the vertex set division above, there are the following adjacent relationships between the vertices: $\mathcal{U}_1 \sim \mathcal{U}_4$, $\mathcal{U}_2 \sim \mathcal{U}_3$, $\mathcal{U}_3 \sim \mathcal{U}_4$ and $\mathcal{U}_4 \sim \mathcal{U}_4$.

Since $\mathcal{U}_4 \sim \mathcal{U}_4$, at least $|\mathcal{U}_4|$ numbers are required. Since $\mathcal{U}_3 \sim \mathcal{U}_4$ and $\mathcal{U}_3 \not\sim \mathcal{U}_3$, all vertices in $\mathcal{U}_1$ and $\mathcal{U}_3$ are labeled with the same number. Since $\mathcal{U}_2 \not\sim \mathcal{U}_4$, the vertices in $\mathcal{U}_2$ are labeled with one of the labels in $\mathcal{U}_4$. Therefore, $\gamma = |\mathcal{U}_4| + 1 = \varrho - 1 + 1 = \varrho$.

From Eq. (1.2),

$$CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2 q})) = \sum_{w \in \mathcal{U}_1} \varsigma(w)^2 + \sum_{w \in \mathcal{U}_2} \varsigma(w)^2 + \sum_{w \in \mathcal{U}_3} \varsigma(w)^2 + \sum_{w \in \mathcal{U}_4} \varsigma(w)^2$$

Since $q > \varrho$, there is $|\mathcal{U}_4| < |\mathcal{U}_3| < |\mathcal{U}_2| < |\mathcal{U}_1|$. Selected $\varsigma(w) = 1$ for $w \in \mathcal{U}_1, \mathcal{U}_3$ and $\varsigma(w) = 2$ for $w \in \mathcal{U}_2$ then

$$\begin{aligned} CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2 q})) &= \sum_{w \in \mathcal{U}_1} 1^2 + \sum_{w \in \mathcal{U}_2} 2^2 + \sum_{w \in \mathcal{U}_3} 1^2 + \sum_{i=2}^{\varrho} i^2 \\ &= (q-1)(\varrho-1) + 4\varrho(\varrho-1) + (q-1) + \left[\frac{\varrho(\varrho+1)(2\varrho+1)}{6} - 1 \right] \\ &= \varrho q + \frac{9}{2}\varrho^2 + \frac{\varrho^3}{3} - \frac{29\varrho}{6} - 1. \end{aligned}$$

Selected $\varsigma(w) = \varrho$ for $w \in \mathcal{U}_1, \mathcal{U}_3$ and $\varsigma(w) = \varrho - 1$ for $w \in \mathcal{U}_2$ then

$$\begin{aligned} CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2 q})) &= \sum_{w \in \mathcal{U}_1} \varrho^2 + \sum_{w \in \mathcal{U}_2} (\varrho-1)^2 + \sum_{w \in \mathcal{U}_3} \varrho^2 + \sum_{i=1}^{\varrho-1} i^2 \\ &= \varrho^2(q-1)(\varrho-1) + \varrho(\varrho-1)(\varrho-1)^2 + \varrho^2(q-1) + \left[\frac{\varrho(\varrho-1)(2\varrho-1)}{6} \right]. \end{aligned}$$

Corollary 2.5. Since $\varsigma(w) = \varrho - 1$ for $w \in \mathcal{U}_1$ and $\varsigma(w) = q - 1$ for $w \in \mathcal{U}_2$, and $\varsigma(w) = (\varrho - 1)(\varrho + 1)$ for $w \in \mathcal{U}_3$, and $\varsigma(w) = (\varrho - 1)(q - 1) + (q - 1) + (\varrho - 1) - 1$ for $w \in \mathcal{U}_4$, the following inequality is obtained:

$$M_1(\Gamma(\mathbb{Z}_{\varrho^2 q})) > CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2 q})) > CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2 q}))$$

where $M_1(\Gamma(\mathbb{Z}_{\varrho^2 q})) = (\varrho - 1)(q - 1)^2(\varrho q - q - \varrho + 1) + (\varrho - 1)((\varrho - 1)^2(q - 1) + (\varrho q - 2)^2)$.

Figure 4 shows plot of $M_1(\Gamma(\mathbb{Z}_{\varrho^2 q}))$, $CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2 q}))$, $CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2 q}))$.

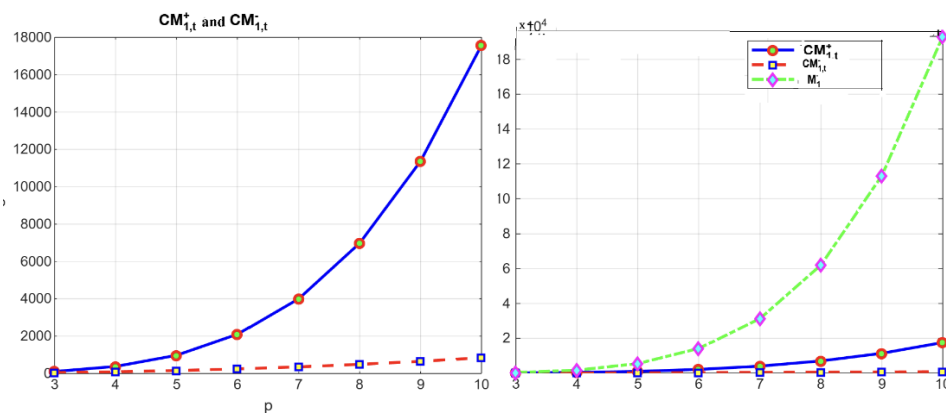


Figure 4. The plot of $M_1(\Gamma(\mathbb{Z}_{\varrho^2 q}))$, $CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2 q}))$, $CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2 q}))$ for $q > \varrho$.

Theorem 2.6. Let $\Gamma(\mathbb{Z}_{\varrho^2 q^2})$ be graph of $\mathbb{Z}_{\varrho^2 q^2}$ non-zero zero divisor rings. If $q < \varrho$ then,

$$CM_{1,t}^+(\Gamma(\mathbb{Z}_{\varrho^2 q^2})) = \varrho(q-1)(\varrho q-1)^2(q+\varrho-1) + 4\varrho(\varrho-1)(\varrho q-2)^2 + \frac{\varrho q(\varrho q-1)(2\varrho q-1)}{6},$$

$$CM_{1,t}^-(\Gamma(\mathbb{Z}_{\varrho^2 q^2})) = \varrho q(q+\varrho-2) + 3\varrho(\varrho-1) + \frac{\varrho q(\varrho q-1)(2\varrho q-1)}{6}.$$

Proof. $V(\Gamma(\mathbb{Z}_{q^2q^2}))$ can be divided:

$$\mathcal{U}_1 = \{\varrho\lambda | 1 \leq \lambda \leq \varrho q^2 - 1, \varrho \nmid \lambda, q \nmid \lambda\}, |\mathcal{U}_1| = q(q-1)(\varrho-1)$$

$$\mathcal{U}_2 = \{q\lambda | 1 \leq \lambda \leq \varrho^2 q - 1, \varrho \nmid \lambda, q \nmid \lambda\}, |\mathcal{U}_2| = \varrho(\varrho-1)(q-1)$$

$$\mathcal{U}_3 = \{\varrho^2\lambda | 1 \leq \lambda \leq q^2 - 1, \varrho \nmid \lambda\}, |\mathcal{U}_3| = q(q-1)$$

$$\mathcal{U}_4 = \{q^2\lambda | 1 \leq \lambda \leq \varrho^2 - 1, q \nmid \lambda\}, |\mathcal{U}_4| = \varrho(\varrho-1)$$

$$\mathcal{U}_5 = \{\varrho q\lambda | 1 \leq \lambda \leq \varrho q - 1, \varrho \nmid \lambda, q \nmid \lambda\}, |\mathcal{U}_5| = (q-1)(\varrho-1)$$

$$\mathcal{U}_6 = \{\varrho^2 q\lambda | 1 \leq \lambda \leq q - 1, q \nmid \lambda\}, |\mathcal{U}_6| = (q-1)$$

$$\mathcal{U}_7 = \{\varrho q^2\lambda | 1 \leq \lambda \leq \varrho - 1, q \nmid \lambda\}, |\mathcal{U}_7| = (\varrho-1)$$

There are relations $|\mathcal{U}_7| < |\mathcal{U}_6| < |\mathcal{U}_4| < |\mathcal{U}_5| < |\mathcal{U}_3| < |\mathcal{U}_2| < |\mathcal{U}_1|$ since $q > \varrho$. When $\varrho = 2$ and $q = 3$, $|\mathcal{U}_4| = |\mathcal{U}_5|$. As can be seen from the vertex set division above, there are the following adjacent relationships between the vertices: $\mathcal{U}_1 \sim \mathcal{U}_7$, $\mathcal{U}_2 \sim \mathcal{U}_6$, $\mathcal{U}_3 \sim \mathcal{U}_4$, $\mathcal{U}_3 \sim \mathcal{U}_7$, $\mathcal{U}_4 \sim \mathcal{U}_6$, $\mathcal{U}_5 \sim \mathcal{U}_5$, $\mathcal{U}_5 \sim \mathcal{U}_6$, $\mathcal{U}_5 \sim \mathcal{U}_7$, $\mathcal{U}_6 \sim \mathcal{U}_6$, $\mathcal{U}_6 \sim \mathcal{U}_7$ and $\mathcal{U}_7 \sim \mathcal{U}_7$. Since $\mathcal{U}_5 \sim \mathcal{U}_5$, $\mathcal{U}_6 \sim \mathcal{U}_6$, $\mathcal{U}_7 \sim \mathcal{U}_7$, and the vertices of these three sets are adjacent to each other, at least $|\mathcal{U}_5| + |\mathcal{U}_6| + |\mathcal{U}_7|$ numbers are required. The vertices in $\mathcal{U}_1$ can be labeled with one of the labels in $\mathcal{U}_5$ and $\mathcal{U}_6$. The vertices in $\mathcal{U}_2$ can be labeled with one of the labels in $\mathcal{U}_5$ or $\mathcal{U}_7$. The vertices in $\mathcal{U}_3$ can be labeled with one of the labels in $\mathcal{U}_5$ or $\mathcal{U}_6$. $\mathcal{U}_4$ can be labeled with one of the labels in $\mathcal{U}_5$ or $\mathcal{U}_7$. Then, $\gamma = (\varrho-1)(q-1) + \varrho-1 + q-1 = \varrho q - 1$.

For minimum, $\varsigma(w) = 1$ is chosen when $w \in \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$. The labeling of the vertices in $\mathcal{U}_5$, which has the most vertices compared to $\mathcal{U}_6$ and $\mathcal{U}_7$, starts from 1. $\varsigma(w) = 2$ is chosen when $w \in \mathcal{U}_4$. From Eq. (1.2),

$$\begin{aligned} CM_{1,t}^{-}(\Gamma(\mathbb{Z}_{q^2q^2})) &= \sum_{w \in \mathcal{U}_1} 1^2 + \sum_{w \in \mathcal{U}_2} 1^2 + \sum_{w \in \mathcal{U}_3} 1^2 + \sum_{w \in \mathcal{U}_4} 2^2 + \sum_{i=1}^{(\varrho-1)(q-1)} i^2 + \sum_{i=(\varrho-1)(q-1)+1}^{(\varrho q - \varrho)} i^2 \\ &\quad + \sum_{i=\varrho q - \varrho + 1}^{\varrho q - 1} i^2 \\ &= q(\varrho-1)(q-1) + \varrho(\varrho-1)(q-1) + q(q-1) + 4\varrho(\varrho-1) + \sum_{i=1}^{\varrho q - 1} i^2 \\ &= \varrho q(q + \varrho - 2) + 3\varrho(\varrho-1) + \frac{\varrho q(\varrho q - 1)(2\varrho q - 1)}{6}. \end{aligned}$$

For equation to be maximum: $\varsigma(w) = \varrho q - 1$ is chosen when $w \in \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$. $\varsigma(w) = \varrho q - 2$ is chosen when $w \in \mathcal{U}_4$. The labeling of the vertices in $\mathcal{U}_5, \mathcal{U}_6, \mathcal{U}_7$ starts from $\varrho + q$, ϱ , 1, respectively. From (1.3),

$$\begin{aligned}
 CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^2q^2})) &= \sum_{w \in \mathcal{U}_1} (\varrho q - 1)^2 + \sum_{w \in \mathcal{U}_2} (\varrho q - 1)^2 + \sum_{w \in \mathcal{U}_3} (\varrho q - 1)^2 + \sum_{w \in \mathcal{U}_4} (\varrho q - 2)^2 + \sum_{i=(\varrho+q)}^{(\varrho q-1)} i^2 \\
 &+ \sum_{i=\varrho}^{\varrho+q+1} i^2 + \sum_{i=1}^{\varrho-1} i^2 \\
 &= q(\varrho - 1)(q - 1)(\varrho q - 1)^2 + \varrho(\varrho - 1)(q - 1)(\varrho q - 1)^2 + q(q - 1)(\varrho q - 1)^2 \\
 &+ 4\varrho(\varrho - 1)(\varrho q - 2)^2 + \sum_{i=1}^{\varrho q-1} i^2 \\
 &= (q - 1)\varrho(\varrho q - 1)^2(\varrho + q - 1) + 4\varrho(\varrho - 1)(\varrho q - 2)^2 + \frac{\varrho q(\varrho q - 1)(2\varrho q - 1)}{6}
 \end{aligned}$$

Corollary 2.6. Since $\varsigma(w) = \varrho - 1$ is chosen when $w \in \mathcal{U}_1$, $\varsigma(w) = q - 1$ is chosen when $w \in \mathcal{U}_2$, $\varsigma(w) = (\varrho - 1)(\varrho + 1)$ for $w \in \mathcal{U}_3$, $\varsigma(w) = (q - 1)(q + 1)$ for $w \in \mathcal{U}_4$, $\varsigma(w) = (\varrho q - 2)$ for $w \in \mathcal{U}_5$, $\varsigma(w) = q(\varrho - 1)(\varrho + 1) + q - 2$ for $w \in \mathcal{U}_6$, and $\varsigma(w) = \varrho(q - 1)(q + 1) + \varrho - 2$ for $w \in \mathcal{U}_7$, the following inequality is obtained:

$$CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^2q^2})) > M_1(\Gamma(\mathbb{Z}_{q^2q^2})) > CM_{1,t}^-(\Gamma(\mathbb{Z}_{q^2q^2}))$$

Where $M_1(\Gamma(\mathbb{Z}_{q^2q^2})) = (\varrho - 1)^2 q(q - 1)(\varrho^2 + 3\varrho + q(\varrho + 1)^2) + (q - 1)^2 \varrho(\varrho - 1)(q^2 + 3q + \varrho(q + 1)^2) + (\varrho^2 - 1)((\varrho q - 2)^2 + 2q(q - 1)(q - 2)) + (q - 1)(q - 2)^2 + 2\varrho(\varrho - 1)(\varrho - 2)(q^2 - 1) + (\varrho - 1)(\varrho - 2)^2$.

Figure 5 shows plot of $CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^2q^2}))$, $M_1(\Gamma(\mathbb{Z}_{q^2q^2}))$, $CM_{1,t}^-(\Gamma(\mathbb{Z}_{q^2q^2}))$ for $q > \varrho$.

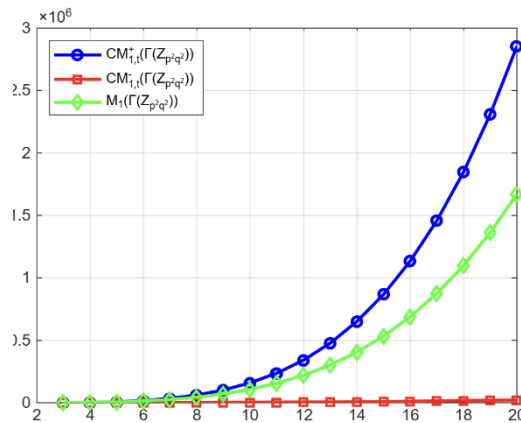


Figure 5. The plot of $M_1(\Gamma(\mathbb{Z}_{q^2q^2}))$, $CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^2q^2}))$, $CM_{1,t}^-(\Gamma(\mathbb{Z}_{q^2q^2}))$.

3. Conclusion

This study evaluates the chromatic first Zagreb indices for zero-divisor graphs derived from commutative rings $\mathbb{Z}_{q^k}$ ($k \geq 2$), $\mathbb{Z}_{\varrho q}$, $\mathbb{Z}_{\varrho^2 q}$, and $\mathbb{Z}_{\varrho^2 q^2}$. It is seen that the first Zagreb index values of the studied algebraic structures are greater than the chromatic first Zagreb index values except for the graphs $\Gamma(\mathbb{Z}_4)$, $\Gamma(\mathbb{Z}_6)$, $\Gamma(\mathbb{Z}_9)$, $\Gamma(\mathbb{Z}_{16})$. In the graphs $\Gamma(\mathbb{Z}_4)$, $\Gamma(\mathbb{Z}_9)$, $\Gamma(\mathbb{Z}_{16})$, the

chromatic first Zagreb indices are greater than the first Zagreb indices and also the minimum and maximum chromatic Zagreb indices are equal. In the graph $\Gamma(\mathbb{Z}_6)$, the first Zagreb index and the minimum chromatic first Zagreb index are equal and less than the maximum chromatic first Zagreb index. The index values of $CM_{1,t}^-$ for all algebraic graphs examined are always smaller than the index values of $CM_{1,t}^+$ and M_1 . For $k \geq 3$, the value of $CM_{1,t}^+(\Gamma(\mathbb{Z}_{q^k}))$ grows faster than the value of $M_1(\Gamma(\mathbb{Z}_{q^k}))$. While M_1 of the graphs $\mathbb{Z}_{q^2q}$, $\mathbb{Z}_{q^2q}$ grows faster than $CM_{1,t}^+$, the opposite is true for the graph $\mathbb{Z}_{q^2q^2}$.

The results can be used in applications in many fields such as cryptology, algorithm analysis and coding theory, by allowing the structural properties of commutative rings and zero divisors to be predicted. It also allows the discovery of the properties of many chemical structures similar to the graphs of $\mathbb{Z}_{q^k}$ ($k \geq 2$), $\mathbb{Z}_{q^2q}$, $\mathbb{Z}_{q^2q}$, and $\mathbb{Z}_{q^2q^2}$ structure

REFERENCE

- [1] Ahmadi MR, Jahani-Nezhad R. Energy and Wiener index of zero-divisor graphs. Iranian Journal of Mathematical Chemistry, 2011, 2(1 (Special Issue on the Occasion of Mircea V. Diudea's Sixtieth Birthday)): 45-51.
- [2] Alali AS, Ali S, Hassan N, Mahnashi AM, Shang Y, Assiry A. Algebraic Structure Graphs over the Commutative Ring $\mathbb{Z}_m$: Exploring Topological Indices and Entropies Using M-Polynomials. Mathematics 2023; 11: 3833. <https://doi.org/10.3390/math11183833>
- [3] Anderson DF, Livingston PS. The zero-divisor graph of a commutative ring. Journal of Algebra, 1999; 217: 434-447.
- [4] Beck I. Coloring of commutative rings, Journal of Algebra 1988; 116: 208-226.
- [5] Cayley, A. Desiderata and suggestions: No.2. The theory of groups: Graphical representation, American Journal of Mathematics, 1878;1(2), 174-176.
- [6] Chartrand G, Zhang P. Chromatic graph theory, (Chapman and Hall/CRC, 2019).
- [7] Dong H, Zhang J, Wang T, Zhang C. Symmetry-aware robot design with structured subgroups. In International Conference on Machine Learning (2023, July), (pp. 8334-8355). PMLR.
- [8] Elahi K, Ahmad A, Asim MA, Hasni, R. Computation of Edge-Based Topological Indices for Zero Divisor Graphs of Commutative Rings, Italian Journal of Pure and Applied Mathematics, 2022; 48: 523-534.
- [9] Gutman I, Trinajstić N. Graph theory and molecular orbitals. Total π -electron energy of alternant hydrocarbons. Chemical physics letters 1972; 17(4): 535-538.
- [10] Kok J, Sudev NK, Mary U. On chromatic Zagreb indices of certain graphs. Discrete Mathematics, Algorithms and Applications, 2017; 9(01): 1750014.

- [11] Kotorowicz JS, Ustimenko VA. On the implementation of cryptoalgorithms based on algebraic graphs over some commutative rings. *Condensed Matter Physics* 2008; 11(2(54)): 347–360.
- [12] Mazlan NA, Mat Hassim HI, Sarmin NH, Salih Khasraw S M. The first Zagreb index of zero-divisor type graph for some rings of integers modulo n . *Proceedings of Science and Mathematics* 2022; (7): 29-32.
- [13] Rayer CJ, Jeyaraj RS. Applications on topological indices of zero-divisor graph associated with commutative rings. *Symmetry* 2023; 15(2): 335.
- [14] Shaska T, Ustimenko V. On the homogeneous algebraic graphs of large girth and their applications. *Linear algebra and its applications*, 2009; 430(7): 1826-1837.
- [15] Semil G, Sarmin NH, Alimon NI, Maulana F. The First Zagreb Index of the Zero Divisor Graph for the Ring of Integers Modulo Power of Primes. *Malaysian Journal of Fundamental and Applied Sciences* 2023; 19(5): 892-900.
- [16] Wiener H. Structural determination of paraffin boiling points, *Journal of the American Chemical Society*, 1947; 69:17-20.



ICSAS

3rd International Conference on Mathematics
September 19 - 21, 2025
Izmir

NO : ICSAS-40A-2025.176S- 1345

Subject: Acceptance letter

16/09/2025

Assistant Professor, ÖZGE ÇOLAKOĞLU
Mersin University,

Herewith, the international scientific committee is happy to inform you that the peer-reviewed draft paper entitled “**CHROMATIC ZAGREB INDEX OF ZERO DIVISOR GRAPHS**” has been accepted for oral presentation as well as inclusion in the conference proceedings of the “ICSAS 3rd International Conference on Mathematics” to be held in Izmir, during September 19 - 21, 2025.

We look forward to receiving your completed registration and seeing you in Izmir.

Sincerely

Prof. Dr. Elif Akpınar Külekçi

ICSAS ACADEMY

CERTIFICATE

OF ACHIEVEMENT

THIS CERTIFICATE IS PRESENTED TO

Assistant Professor, ÖZGE ÇOLAKOĞLU

IN ORAL PRESENTATION, RECOGNITION AND APPRECIATION

OF RESEARCH CONTRIBUTION TO

ICSAS 3rd INTERNATIONAL CONFERENCE ON MATHEMATICS

ON SEPTEMBER 19 - 21, 2025 IN IZMIR

WITH THE PAPER ENTITLED

"CHROMATIC ZAGREB INDEX OF ZERO DIVISOR GRAPHS"



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